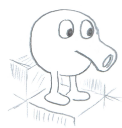
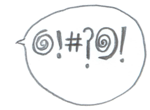
MATH 124: Matrixology (Linear Algebra)


Level Q*bert (1982), 9 of 10
University of Vermont, Spring 2015


Dispersed: Wednesday, April 8, 2015.
Due: By start of lecture, Tuesday, April 21, 2015.
Sections covered: 6.1, 6.2, 6.4, 6.6.

Some useful reminders:
Instructor: Prof. Peter Dodds
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E-mail: peter.dodds@uvm.edu
Office hours: 12:30 to 3:00 pm Mondays
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2015-01UVM-124
Textbook: "Introduction to Linear Algebra" (3rd or 4th edition) by Gilbert Strang (published by Wellesley-Cambridge Press).

- All questions are worth 3 points unless marked otherwise.
- Please use a cover sheet and write your name on the back and the front of your assignment.
- You must show all your work clearly.
- You may use Matlab to check your answers for non-Matlab questions (usually Qs. 1–8).
- Please list the names of other students with whom you collaborated.

1. Diagonalize the Fibonacci matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ to find Λ. Write down S and calculate S^{-1}.

2. Write down a matrix A that does *both* of the following two actions to vectors living in R^2:
   - A stretches vectors that are proportional to $[\ 1\ \ -2\ ]^{\mathrm{T}}$ by a factor of 3.
   - For vectors that are proportional to $[\ 2\ 1\ ]^{\mathrm{T}}$, A shrinks their length to 1/2 original size and makes them point in the opposite direction.

3. Express the vector $\vec{w} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ in terms of the basis
$$\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}.$$

4. (part of Q 20, Section 6.2)

Diagonalize the following matrix

$$A = \begin{bmatrix} .6 & .4 \\ .4 & .6 \end{bmatrix}.$$

Write down S, Λ, and S^{-1}.

What does Λ^n tend towards as $n \to \infty$?

Consequently, what does $A^n = S\Lambda^n S^{-1}$ tend toward?

5. Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

Turn the eigenvectors into unit vectors.

Please order the eigenvalues from most positive to least positive (this is a standard procedure; sorting by magnitude is often needed too. So, if you found the eigenvalues for a 4x4 matrix were -1, 3, 0, and 4, then you would assign them as follows: $\lambda_1 = 4$, $\lambda_1 = 3$, $\lambda_1 = 0$, and $\lambda_1 = -1$.)

6. The spectral theorem for symmetric matrices:

Using your results from the previous question, re-express A as

$$A = \lambda_1 \hat{v}_1 \hat{v}_1^{\mathrm{T}} + \lambda_2 \hat{v}_2 \hat{v}_2^{\mathrm{T}} + \lambda_3 \hat{v}_3 \hat{v}_3^{\mathrm{T}}$$

where $\hat{v}_1$, $\hat{v}_2$, and $\hat{v}_3$ are A's normalized (unit) eigenvectors. This is the *spectral decomposition of* A.

Compute the products involving the eigenvectors (these are outer products) and show that the right hand side of the above equation indeed equals A.

7. Building further on the previous two questions:

   (a) Fill in the blanks: A is a ____ matrix so its eigenvectors are ____.

   (b) Write down the diagonal matrix Λ (we say that Λ is *similar* to A).

   (c) Use the normalized eigenvectors to create Q and Q^{T} (the usual S and S^{-1} become Q and Q^{T} for symmetric matrices). Check that $Q^{\mathrm{T}}Q = I$.

   (d) Does $QQ^{\mathrm{T}} = I$? Why or why not?

8. Compute e^{At} when

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ and when (b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

To do this, we use the Taylor expansion for the exponential:

$$e^x = 1 + x + x^2/2! + \ldots + x^n/n! + \ldots$$

which, for matrices, looks like this:

$$e^A = I + A + A^2/2! + \ldots + A^n/n! + \ldots$$

When we have a t floating around (t is for time), then the expansion is thus:

$$e^{At} = I + At + A^2t^2/2! + \ldots + A^nt^n/n! + \ldots$$

This kind of matrix exponential beastie naturally appears in the study of coupled linear differential equations, which I can imagine is very exciting for you. Mathematically, it's surprising we can do such things.

Anyway, if A can be diagonalized, then $A = S\Lambda S^{-1}$ and the expansion for e^{At} becomes

$$\begin{aligned}
e^{At} &= I + At + A^2t^2/2! + \ldots + A^nt^n/n! + \ldots \\
&= SIS^{-1} + S\Lambda S^{-1}t + (S\Lambda S^{-1})^2t^2/2! + \ldots + (S\Lambda S^{-1})^nt^n/n! + \ldots \\
&= SIS^{-1} + S\Lambda S^{-1}t + S\Lambda^2S^{-1}t^2/2! + \ldots + S\Lambda^nS^{-1}t^n/n! + \ldots \\
&= S(I + \Lambda t + \Lambda^2t^2/2! + \ldots + \Lambda^nt^n/n! + \ldots)S^{-1} \\
&= S \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} S^{-1}.
\end{aligned}$$

So, your task is to employ this formula for the two matrices given above (please digest its derivation as well).

Hint: (a) for the first matrix, $S = S^{-1} = I$.

9. Matlab question:

   Use Matlab to compute the exponentials of the matrices given in the previous question for $t = 1$. Check your formulas from question 8 match.

```
>> A = [ 1 2 ; 2 1 ];
>> expm(A)
```

   Note: please use `expm`, not `exp`.

10. Matlab question.

    Use Matlab to compute the eigenvalues and eigenvectors for $\mathbf{A}$ given in question 5. Check you obtain the same as your pencil and paper calculations.

11. (Bonus question, 1 point)

    Based on venom levels, which Australian organism should ophidiophobics be most afraid of?