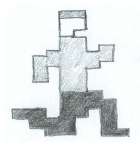
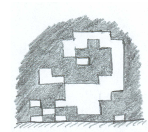


MATH 124: Matrixology (Linear Algebra)
Level Pitfall! (1982), 8 of 10
University of Vermont, Spring 2015



Dispersed: Saturday, April 4, 2015.
Due: By start of lecture, Tuesday, April 14, 2015.
Sections covered: 5.1—5.3, 6.1, a touch of 6.2.

Some useful reminders:
Instructor: Prof. Peter Dodds
Office: Farrell Hall, second floor, Trinity Campus
E-mail: peter.dodds@uvm.edu
Office hours: 12:30 to 3:00 pm Mondays
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2015-01UVM-124
Textbook: "Introduction to Linear Algebra" (3rd or 4th edition) by Gilbert Strang (published by Wellesley-Cambridge Press).

- All questions are worth 3 points unless marked otherwise.
- Please use a cover sheet and write your name on the back and the front of your assignment.
- You must show all your work clearly.
- You may use Matlab to check your answers for non-Matlab questions (usually Qs. 1–8).
- Please list the names of other students with whom you collaborated.

Note: Most of what you need to know to get through this level is covered in Episodes 19a, 19b, and 19c: http://t.co/McJaSLuTzI

1. Using row operations, turn the following two determinants into some number multiplying the determinant of the identity matrix $\mathbf{I}$. Remember that we showed that the normal row operation of adding a multiple of one row to another does not change the value of a determinant. However, if you divide a row or column by a number, you have to keep this number out the front (multilinearity), and if you swap two rows or swap two columns, the determinant's sign changes.

   (a) $\begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix}$ and (b) $\begin{vmatrix} 0 & 2 \\ 3 & 4 \end{vmatrix}$

2. (Q 14, 5.1) Use row operations to find $\mathbf{U}$ and therefore the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}$$

3. (Q 24, 5.1) Back to our first factorization (of which we have such fond memories): $\mathbf{A} = \mathbf{LU}$. Consider

$$\mathbf{A} = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \mathbf{LU}.$$

Find the determinants of (a) $\mathbf{L}$, (b) $\mathbf{U}$, (c) $\mathbf{A}$, (d) $\mathbf{U}^{-1}$, (e) $\mathbf{L}^{-1}$, and (f) $\mathbf{U}^{-1}\mathbf{L}^{-1}\mathbf{A}$ (what is this matrix?).

4. This question is about finding determinants through cofactors and capitalizing on the 0's present in the matrix.

Help—S8E19b: Computing determinants using the Way of the Cofactor (25:49): `http://www.youtube.com/v/ln-bY7_vAPA?rel=0`

Notes are here: http://www.uvm.edu/∼pdodds/teaching/courses/2015-01UVM-124/docs/2015-04-03cofactors.pdf

For the matrix given above in question 2,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix},$$

find the following:

(a) The minor matrices $\mathbf{M}_{42}$ and $\mathbf{M}_{44}$

(b) The cofactors C_{42} and C_{44} (write down how cofactors are connected to the minor matrices). (Hint: when computing $|\mathbf{M}_{42}|$ and $|\mathbf{M}_{44}|$, 'go along' the row or column with the most 0's).

(c) Hence find the determinant of $\mathbf{A}$ by 'going along' the bottom row.

(Note that going along the third row or third column would have been equally good choices.)

5. (Q1, Section 5.3)

Use Cramer's rule to solve

$$\begin{aligned} 2x_1 + 5x_2 &= 1 \\ x_1 + 4x_2 &= 2 \end{aligned}$$

Recall the rule for solving $\mathbf{A}\vec{x} = \vec{b}$ when $\mathbf{A}$ is square (n by n) and $|\mathbf{A}| \neq 0$:

$$\vec{x} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} |\mathbf{B}_1| \\ |\mathbf{B}_2| \\ \vdots \\ |\mathbf{B}_n| \end{bmatrix}$$

where $\mathbf{B}_i$ is $\mathbf{A}$ with the ith column replaced by $\vec{b}$.

A complete explanation of Cramer's rule is here: **Help—Cramer's rule** (17:02): `http://www.youtube.com/v/iw_fqOO-fN8?rel=0`

Notes are here: http://www.uvm.edu/∼pdodds/teaching/courses/2015-01UVM-124/docs/2015-04-03cramers-rule.pdf

6. You too can experience the unbridled excitement of finding inverses through cofactors:

   (a) Find the cofactor matrix $\mathbf{C}$ of the general 2 by 2:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

   Then find the form of the inverse: $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{C}^{\mathrm{T}}$.

   (b) Do the same for

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

   An explanation for this procedure starts part way into the Cramer's Rule episode: **Help—Cramer's rule** (17:02): `http://www.youtube.com/v/iw_fqOO-fN8?rel=0`

   Notes are here: http://www.uvm.edu/∼pdodds/teaching/courses/2015-01UVM-124/docs/2015-04-03cramers-rule.pdf

7. Find the eigenvalues and eigenvectors for the following matrices. Write down the algebraic multiplicities and geometric multiplicities for each eigenvalue.

   Steps: Solve $|\mathbf{A} - \lambda\mathbf{I}| = 0$ to find the eigenvalues, then solve $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$ to find their eigenvectors (really spaces).

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

   Everything you need to know about multiplicities explained in the webisode below. Brief reminder: The number of times an eigenvalue λ is repeated (as a solution to the characteristic equation) is its algebraic multiplicity, and the number of linearly independent eigenvectors associated with λ is its geometric multiplicity.

   Help—Algebraic and Geometric Multiplicity of Eigenvectors (9:26): `http://www.youtube.com/v/Lu6CJi7z-ls?rel=0`

8. Show that any real, symmetric 2 by 2 matrix has real eigenvalues. In other words, your mission is to find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

(where a, b, and c are all real), and show that the eigenvalues are real numbers.

(The formula for the solutions to a quadratic equation will be your friend here. It's always been your friend but maybe you've drifted apart and really it was just a simple misunderstanding. You know how these things go. Right now, it will be very friendly.)

9. Matlab question:

   (a) For the matrix given in question 2,

   $$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix},$$

   use Matlab to find the determinant.

   (b) Find the determinant of $\frac{1}{2}\mathbf{A}$.

   (c) Find the determinant of $\mathbf{A}^{-1}$.

   Note how all values are what you should expect to find.

10. Matlab question:

    Find the inverse of $\mathbf{A}$ given in question 2 by using the cofactor approach in question 6. Use Matlab to determine the 16 elements in the cofactor matrix and form the inverse according to our formula:

    $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{C}^{\mathrm{T}}$.

11. (Bonus, 1 point) What's the name of the largest species of kangaroo and what's a rough upper bound on their height?