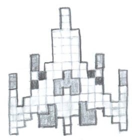

MATH 124: Matrixology (Linear Algebra)
Level Galaga (1981), 7 of 10
University of Vermont, Spring 2015

Dispersed: Saturday, March 21, 2015.
Due: By start of lecture, Tuesday, April 7, 2015.
Sections covered: 4.4, 6.1, a smidgeon of 5.1.

Some useful reminders:
Instructor: Prof. Peter Dodds
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E-mail: peter.dodds@uvm.edu
Office hours: 12:30 to 3:00 pm Mondays
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2015-01UVM-124
Textbook: "Introduction to Linear Algebra" (3rd or 4th edition) by Gilbert Strang (published by Wellesley-Cambridge Press).

- All questions are worth 3 points unless marked otherwise.
- Please use a cover sheet and write your name on the back and the front of your assignment.
- You must show all your work clearly.
- You may use Matlab to check your answers for non-Matlab questions (usually Qs. 1–8).
- Please list the names of other students with whom you collaborated.

1. (Q 5, Section 4.4) Find two orthogonal vectors in the plane $x_1 + x_2 + 2x_3 = 0$. Make these vectors into an orthonormal basis.

   Do this by (a) finding a basis for the null space of the matrix $\mathbf{A} = [\,1\ 1\ 2\,]$ and (b) using the Gram-Schmidt Process to generate an orthogonal basis.

2. First, please absorb this short video:

   Help—Definition of Orthogonal matrices (6:36):
   `http://www.youtube.com/v/hQG6w7Myqtw?rel=0`

   Now show that any orthogonal matrix $\mathbf{Q}$ preserves lengths and angles when it transforms vectors. Remember that orthogonal matrices have the special property that $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$ (also: let's allow $\mathbf{Q}$ to be an m by n matrix where $m \geq n$—$\mathbf{Q}$ need not be square for this question).

   (a) Lengths: Show that $\mathbf{Q}\vec{x}$ has the same length (magnitude) as $\vec{x}$.

   Hint—Showing that the squares of the lengths match is sufficient.

   (b) Angles: Show that the cosine of the angle between two vectors $\vec{x}$ and $\vec{y}$ is the same as that for the angle between $\mathbf{Q}\vec{x}$ and $\mathbf{Q}\vec{y}$.

Hint—We can show that the cosine of the angle between the $\vec{x}$ and $\vec{y}$ is the same as the cosine of the angle between $\mathbf{Q}\vec{x}$ and $\mathbf{Q}\vec{y}$. By rearranging the dot product formula,

$$\vec{x}^{\mathrm{T}}\vec{y} = ||\vec{x}||\,||\vec{y}||\cos\theta,$$

where θ is the angle between $\vec{x}$ and $\vec{y}$, we have

$$\cos\theta = \frac{\vec{x}^{\mathrm{T}}\vec{y}}{||\vec{x}||\,||\vec{y}||}.$$

From part (a), we already have that the bottom of this fraction doesn't change ($||\mathbf{Q}\vec{x}|| = ||\vec{x}||$ and $||\mathbf{Q}\vec{y}|| = ||\vec{y}||$), so we need to just show that the numerator remains the same.

3. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -1 & 1 \end{bmatrix},$$

and its corresponding $\mathbf{Q}$ (determined by Gram-Schmidt)

$$\mathbf{Q} = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 0 & 1/3 \\ -1/\sqrt{2} & 2/3 \end{bmatrix},$$

find the upper triangular matrix $\mathbf{R}$ so that $\mathbf{A} = \mathbf{QR}$. (Look up the formula for $\mathbf{R}$).

4. (Q 5, Section 6.1) Find the eigenvalues and eigenvectors of $\mathbf{A}$ and $\mathbf{A}^2$:

$$\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \text{ and } \mathbf{A}^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

How are the eigenvectors and eigenvalues of $\mathbf{A}$ and $\mathbf{A}^2$ related?

5. Compute the eigenvalues (but not the eigenvectors) of $\mathbf{A}^{-1}$ for the $\mathbf{A}$ given above.

How are the eigenvalues of $\mathbf{A}$ and $\mathbf{A}^{-1}$ related?

6. Find the eigenvalues (but not the eigenvectors) of $\mathbf{A} + 3\mathbf{I}$ where for $\mathbf{A}$ as given in question 4.

Now what's the relationship between the eigenvalues of $\mathbf{A}$ and $\mathbf{A} + 3\mathbf{I}$?

7. Building on your findings in the previous questions:

(a) Show that if $\mathbf{A}^{-1}$ exists, then the eigenvalues of $\mathbf{A}^{-1}$ are inverses of the eigenvalues of $\mathbf{A}$, and that they share the same eigenvectors.

(b) Show that λ^2 is an eigenvalue of $\mathbf{A}^2$ if λ is an eigenvalue of $\mathbf{A}$, and that they share the same eigenvectors.

(c) Show that $\lambda + k$ is an eigenvalue of $\mathbf{A} + k\mathbf{I}$ (where k is any number and $\mathbf{I}$ is the identity matrix) if λ is an eigenvalue of $\mathbf{A}$, and that they share the same eigenvectors.

Note: You are deriving these results for arbitrary n by n matrices.

Hint—please see this tweet:
https://twitter.com/matrixologyvox/status/582724638561693696

8. (Q 2, 5.1)

   Watch the following video for help on this question and determinants in general.

   Help—Determinants from the ground up (19:23):
   `http://www.youtube.com/v/CuFUXcXZRq8?rel=0`

   Given a 3 by 3 matrix $\mathbf{A}$ has determinant equal to 5, find (a) det (1/2 $\mathbf{A}$), (b) det(-$\mathbf{A}$), (c) det($\mathbf{A}^2$), and (d) det ($\mathbf{A}^{-1}$).

   Here are some things you are allowed to know (even if we haven't covered them in class yet): $|t\mathbf{A}| = t^n|\mathbf{A}|$ and $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ and $|\mathbf{I}| = 1$.

9. Matlab question:

   Use Matlab's eig command to find the eigenvalues and eigenvectors of the matrix we studied above:

$$\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}.$$

   Basic usage:

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[V,Lambda] = eig(A)
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   The unit eigenvectors are the columns of V and the eigenvalues are on the main diagonal of Lambda.

   Confirm that you have a match with your pencil and paper calculations.

10. Matlab question:

    In class, we talked about a random texter, lost on a network of paths connecting five locations labelled 1–5.

    We found the following transition matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 & 1 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \end{bmatrix},$$

    which connects the probability our magic rectangle user is at location i at time t, $x_{i,t}$, via

$$\vec{x}_{t+1} = \mathbf{A}\vec{x}_t,$$

    where $[\vec{x}_t]_i = x_{i,t}$.

(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$ using Matlab.

(b) Let's start our texter at location 3 at time $t = 0$:

$$\vec{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Using Matlab to compute large powers, what does $\vec{x}_t$ tend toward as $t \to \infty$? Does it matter where our texter starts?

(c) Given $\mathbf{A}$'s eigenvalues, explain why or why not $\mathbf{A}$ is invertible.

11. (Bonus question, 1 point) How many players does each side have on the field in an Australian Rules Football match, and how long is a typical ground (playing area)?