Principles of Complex Systems, CSYS/MATH 300
University of Vermont, Fall 2013
Assignment 5 • code name: Infinite Improbability Drive (⊞)

Dispersed: Thursday, October 3, 2013.
Due: By start of lecture, 1:00 pm, Thursday, October 10, 2013.
Some useful reminders:
Instructor: Peter Dodds
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Office hours: 10:30 am to 11:30 am, Monday, and 1:00 pm to 3:00 pm, Wednesday
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2013-08UVM-300

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related TeX variant).

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1. (3 + 3)

   Coding, it's what's for breakfast:

   (a) Percolation in two dimensions (2-d) provides a classic, nutritious example of a phase transition.

   Your mission, whether or not you choose to accept it, is to code up and analyse the L by L square lattice percolation model for varying L.

   Take $L = 20$, 50, 100, 200, 500, and 1000.

   (Go higher if you feel $L = 1000$ is for mere mortals.)

   (Go lower if your code explodes.)

   Let's continue with the tree obsession. A site has a tree with probability p, and a sheep grazing on what's left of a tree with probability $1 - p$.

   Forests are defined as any connected component of trees bordered by sheep, where connections are possible with a site's four nearest neighbors on a lattice.

   Do not bagelize (or doughnutize) the landscape (no periodic boundary conditions—boundaries are boundaries).

   (Note: this set up is called site percolation. Bond percolation is the alternate case when all links between neighboring sites exist with probability p.)

   Steps:

i. For each L, run $N_{\rm tests}$=100 tests for occupation probability p moving from 0 to 1 in increments of 10^{-2}. (As for L, use a smaller increment if that's just how you do things.)

ii. Determine the fractional size of the largest connected forest for each of the $N_{\rm tests}$, and find the average of these, $S_{\rm avg}$.

iii. On a single figure, for each L, plot the average $S_{\rm avg}$ as a function of p.

(b) Comment on how $S_{\rm avg}(p; N)$ changes as a function of L and estimate the critical probability p_c (the percolation threshold).

Helpful reuse of code (intended for black and white image analysis): You can use Matlab's bwconncomp to find the sizes of components. Very nice.

2. (3 + 3)

(a) Using your model from the previous question and your estimate of p_c, plot the distribution of forest sizes for $p \simeq p_c$ for the largest L your code and psychological makeup can withstand. (You can average the distribution over separate simulations.)

Comment on what kind of distribution you find.

(b) Repeat the above for $p = p_c/2$ and $p = p_c + (1 - p_c)/2$, i.e., well below and well above p_c.

Produce plots for both cases, and again, comment on what you find.

3. Show analytically that the critical probability for site percolation on a triangular lattice is $p_c = 1/2$.
Hint—Real-space renormalization gets it done.:

Direct link: `http://www.youtube.com/v/JlkbU5U7QqU?rel=0`

4. In lectures on lognormals and other heavy-tailed distributions, we came across a super fun and interesting integral when considering organization size distributions arising from growth processes with variable lifespans.

   Show that

$$P(x) = \int_{t=0}^{\infty} \lambda e^{-\lambda t} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln \frac{x}{m})^2}{2t}\right) \mathrm{d}t$$

   leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln \frac{x}{m})^2}},$$

   and therefore, surprisingly, two different scaling regimes. Enjoyable suffering may be involved. Really enjoyable suffering.

   Hints and steps:

   - Make the substitution $t = u^2$ to find an integral of the form

$$I_1(a, b) = \int_0^{\infty} \exp\left(-au^2 - b/u^2\right) \mathrm{d}u$$

     where in our case $a = \lambda$ and $b = (\ln \frac{x}{m})^2/2$.
   - Substitute $au^2 = t^2$ into the above to find

$$I_1(a, b) = \frac{1}{\sqrt{a}} \int_0^{\infty} \exp\left(-t^2 - ab/t^2\right) \mathrm{d}t$$

   - Now work on this integral:

$$I_2(r) = \int_0^{\infty} \exp\left(-t^2 - r/t^2\right) \mathrm{d}t$$

     where $r = ab$.
   - Differentiate I_2 with respect to r to create a simple differential equation for I_2 You will need to use the substitution $u = \sqrt{r}/t$ and your differential equation should be of the form

$$\frac{\mathrm{d}I_2(r)}{\mathrm{d}r} = -(\text{something}) I_2(r).$$

   - Solve the differential equation you find. To find the constant of integration, you can evaluate $I_2(0)$ separately:

$$I_2(0) = \int_0^{\infty} \exp(-t^2) \mathrm{d}t,$$

     where our friend $\Gamma(1/2)$ comes into play.

5. (3 + 3 + 3 + 3) This question is all about pure finite and infinite random networks

   We'll define a finite random network as follows. Take N labelled nodes and add links between each pair of nodes with probability p.

   (a) i. For a random node i, determine the probability distribution for its number of friends k, $P_k(p, N)$.

      ii. What kind of distribution is this?

      iii. What does this distribution tend toward in the limit of large N, if p is fixed?
      (No need to do calculations here; just invoke the right Rule of the Universe.)

   (b) Using $P_k(p, N)$, determine the average degree. Does your answer seem right intuitively?

   (c) Show that in the limit of $N \to \infty$ but with mean held constant, we obtain a Poisson degree distribution.

      Hint: to keep the mean constant, you will need to change p.

   (d) i. Compute the clustering coefficients C_1 and C_2 for standard random networks.

      ii. Explain how your answers make sense.

      iii. What happens in the limit of an infinite random network with finite mean?