## Mechanisms for Generating Power-Law Size Distributions I

Principles of Complex Systems CSYS/MATH 300, Spring, 2013

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## Outline

## Random Walks

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## References



## Mechanisms：

A powerful story in the rise of complexity：
－structure arises out of randomness．
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- Exhibit A: Random walks. ( $\boxplus$ )



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The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x=0$.
- Step at time $t$ is $\epsilon_{t}$ :

1 with probability $1 / 2$
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－Time and space are discrete
－Random walker（e．g．，a drunk）starts at origin $x=0$ ．
－Step at time $t$ is $\epsilon_{t}$ ：

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\epsilon_{t}= \begin{cases}+1 & \text { with probability } 1 / 2 \\ -1 & \text { with probability } 1 / 2\end{cases}
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$\left|\begin{array}{l}0 \\ 0 \\ 0\end{array}\right|$

## A few random random walks:





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## Displacement after $t$ steps：

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- Obviously fails for odd number of steps...
- But as time goes on, the chance of our drunkard lurching back to the pub must diminish, right?



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\operatorname{Var}\left(x_{t}\right)=\operatorname{Var}\left(\sum_{i=1}^{t} \epsilon_{i}\right)
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＊Sum rule＝a good reason for using the variance to measure spread；only works for independent distributions．

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Great moments in Televised Random Walks:


Plinko! (田) from the Price is Right.

## Random walk basics:

## Counting random walks:

- Each specific random walk of length $t$ appears with a chance $1 / 2^{t}$.
- We'll be more interested in how many random walks end up at the same place.
- Define $N(i, j, t)$ as \# distinct walks that start at $x=i$ and end at $x=j$ after $t$ time steps.
- Random walk must displace by $+(j-i)$ after $t$ steps.

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N(i, j, t)=\binom{t}{(t+j-i) / 2}
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How does $P\left(x_{t}\right)$ behave for large $t$ ?

- Take time $t=2 n$ to help ourselves.
- $x_{2 n} \in\{0, \pm 2, \pm 4, \ldots, \pm 2 n\}$
- $x_{2 n}$ is even so set $x_{2 n}=2 k$.
- Using our expression $N(i, j, t)$ with $i=0, j=2 k$, and $t=2 n$, we have

$$
\operatorname{Pr}\left(x_{2 n} \equiv 2 k\right) \propto\binom{2 n}{n+k}
$$

- For large $n$, the binomial deliciously approaches the Normal Distribution of Snoredom:

$$
\operatorname{Pr}\left(x_{t} \equiv x\right) \simeq \frac{1}{\sqrt{2 \pi t}} e^{-\frac{x^{2}}{2 t}} .
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- The whole is different from the parts.


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－See also：Stable Distributions（ $⿴ 囗 十 ⺝)$

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- See also:
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- See also: Stable Distributions ( $\boxplus$ )



## Universality $(\boxplus)$ is also not left-handed:



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- This is Diffusion $(\boxplus)$ : the most essential kind of spreading (more later).
- View as Random Additive Growth Mechanism.


## Random walks are even weirder than you might think...

- $\xi_{r, t}=$ the probability that by time step $t$, a random walk has crossed the origin $r$ times.
- Think of a coin flin game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...
> In fact: $\xi_{0, t}>\xi_{1, t}>\xi_{2, t}$
- Even crazier:

The expected time between tied scores $=\infty$ !

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The expected time between tied scores $=\infty$ ！
See Feller，Intro to Probability Theory，Volume I ${ }^{[3]}$

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## Random walks \#crazytownbananapants

## The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after $t$ steps?
- Will our drunkard always return to the origin?

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## Reasons for caring:



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Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.
We'll start to see how different scalings relate to each other.

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## Random walks \#crazytownbananapants

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after $t$ steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?


## Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent.
2. Some physical structures may result from random walks.

3. We'll start to see how different scalings relate to each other.

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$\left\lvert\, \begin{gathered}0 \\ 6 \\ 0\end{gathered}\right.$

Power－Law
－Idea：Transform first return problem into an easier return problem．
For random walks in 1－d：

$\rightarrow$ A return to origin can only happen when $t=2 n$ ．
－In example above，returns occur at $t=8,10$ ，and 14 ．
－Call $P_{f r}(2 n)$ the probability of first return at $t=2 n$ ．
－Probability calculation $\equiv$ Counting problem （combinatorics／statistical mechanics）．

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For random walks in 1-d:


- A return to origin can only happen when $t=2 n$.
- In example above, returns occur at $t=8,10$, and 14 .
- Call $P_{\mathrm{fr}}(2 n)$ the probability of first return at $t=2 n$.
- Probability calculation $\equiv$ Counting problem (combinatorics/statistical mechanics).
- Idea: Transform first return problem into an easier return problem.

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- Can assume drunkard first lurches to $x=1$.
- Observe walk first returning at $t=16$ stays at or above $x=1$ for $1 \leq t \leq 15$ (dashed red line).
- Now want walks that can return many times to $x=1$
$\Rightarrow P_{\mathrm{fr}}(2 n)=$ $2 \cdot \frac{1}{2} \operatorname{Pr}\left(x_{t} \geq 1,1 \leq t \leq 2 n-1\right.$, and $\left.x_{1}=x_{2 n-1}=1\right)$
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- The 2 accounts for drunkards that first lurch to

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## Counting first returns:

## Approach:

- Move to counting numbers of walks.
- Return to probability at end.
- Again, $N(i, j, t)$ is the \# of possible walks between $x=i$ and $x=j$ taking $t$ steps.
- Consider all paths starting at $x=1$ and ending at $x=1$ after $t=2 n-2$ steps.
- Idea: If we can compute the number of walks that hit $x=0$ at least once, then we can subtract this from the total number to find the ones that maintain $x \geq 1$.
- Call walks that drop below $x=1$ excluded walks.
- We'll use a method of images to identify these excluded walks.

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## Examples of excluded walks：




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Key observation for excluded walks：
－For any path starting at $x=1$ that hits 0 ，there is a unique matching path starting at $x=-1$ ．
－Matching path first mirrors and then tracks after first reaching $x=0$ ．
－\＃of $t$－step paths starting and ending at $x=1$ and hitting $x=0$ at least once

$\left|\begin{array}{l}0 \\ 5 \\ 0\end{array}\right|$ $\rightarrow$ So $N_{\text {first return }}(2 n)=N(1,1,2 n-2)-N(-1,1,2 n-2)$

## Examples of excluded walks:



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- For any path starting at $x=1$ that hits 0 , there is a unique matching path starting at $x=-1$.
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- So $N_{\text {first return }}(2 n)=N(1,1,2 n-2)-N(-1,1,2 n-2)$


## Probability of first return:

## Insert question from assignment 2 ( $\boxplus$ ) :

- Find

$$
N_{\mathrm{fr}}(2 n) \sim \frac{2^{2 n-3 / 2}}{\sqrt{2 \pi} n^{3 / 2}}
$$

- Normalized number of paths gives probability.

$$
P_{\mathrm{fr}}(2 n)=\frac{1}{2^{2 n}} N_{\mathrm{fr}}(2 n)
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## First Returns

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- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin

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P(t) \propto t^{-3 / 2}, \gamma=3 / 2
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- Same scaling holds for continuous space/time walks.
- $P(t)$ is normalizable.
- Recurrence: Random walker always returns to origin
- But mean, variance, and all higher moments are infinite.
\#totalmadness
- Even though walker must return, expect a long wait...
- One moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.


## Higher dimensions ( $\boxplus$ ):

- Walker in $d=2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions

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## Random walks

## On finite spaces:

- In any finite homogeneous space, a random walker will visit every site with equal probability
- Call this probability the Invariant Density of a dynamical system

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## On networks:



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## Scheidegger Networks ${ }^{[8,2]}$

- Random directed network on triangular lattice.
- Toy model of real networks.
- 'Flow' is southeast or southwest with equal probability.



## Scheidegger networks

- Creates basins with random walk boundaries.
- Observe that subtracting one random walk from another gives random walk with increments:
- Random walk with probabilistic pauses.
- Basin termination = first return random walk problem.
- Basin length $\ell$ distribution: $P(\ell) \propto \ell^{-3 / 2}$
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\epsilon_{t}=\left\{\begin{array}{cl}
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## Connections between exponents:

- For a basin of length $\ell$, width $\propto \ell^{1 / 2}$


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## Connections between exponents:

- Both basin area and length obey power law distributions
- Observed for real river networks
- Reportedly: $1.3<\tau<1.5$ and $1.5<\gamma<2$


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\ell \propto a^{h} .
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－For real，large networks $h \simeq 0.5$
－Smaller basins possibly $h>1 / 2$（later：allometry）．
－Models exist with interesting values of $h$ ．
－Plan：Redo calc with $\gamma, \tau$ ，and $h$ ．

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## Connections between exponents:

- Given

```
        \ell\propto a}\mp@subsup{a}{}{h},P(a)\propto\mp@subsup{a}{}{-\tau},\mathrm{ and }P(\ell)\propto\mp@subsup{\ell}{}{-\gamma
    dl }\propto\textrm{d}(\mp@subsup{a}{}{h})=h\mp@subsup{a}{}{h-1}\textrm{d}
- Find }\tau\mathrm{ in terms of }\gamma\mathrm{ and }h\mathrm{ .
- Pr(basin area = a)da
= Pr(basin length = \ell)d
```

$$
\tau=1+h(\gamma-1)
$$

- Excellent example of the Scaling Relations found between exponents describing power laws for many systems.


## Connections between exponents:

- Given

$$
\ell \propto a^{h}, P(a) \propto a^{-\tau}, \text { and } P(\ell) \propto \ell^{-\gamma}
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$\Rightarrow \mathrm{d} \ell \propto \mathrm{d}\left(\mathrm{a}^{h}\right)=h \mathrm{a}^{h-1} \mathrm{da}$

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With more detailed description of network structure, $\tau=1+h(\gamma-1)$ simplifies to: ${ }^{[1]}$

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\tau=2-h
$$

and

$$
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- Only one exponent is independent (take h).
- Simplifies system description.
- Expect Scaling Relations where power laws are found.
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With more detailed description of network structure, $\tau=1+h(\gamma-1)$ simplifies to: ${ }^{[1]}$

$$
\tau=2-h
$$

and

$$
\gamma=1 / h
$$

- Only one exponent is independent (take $h$ ).
- Simplifies system description.
- Expect Scaling Relations where power laws are found.
- Need only characterize
independent exponents.
class with



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- Only one exponent is independent (take $h$ ).
- Simplifies system description.
- Expect Scaling Relations where power laws are found.
- Need only characterize Universality ( $\boxplus$ ) class with independent exponents.



## Other First Returns or First Passage Times:

## Failure:

- A very simple model of failure/death: ${ }^{[10]}$
- $x_{t}=$ entity's 'health' at time $t$
- Start with $x_{0}>0$.
- Entity fails when $x$ hits 0 .


## Streams <br> - Dispersion of suspended sediments in streams. <br> - Long times for clearing.


$\left|\begin{array}{l}0 \\ 0 \\ 0\end{array}\right|$

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## Other First Returns or First Passage Times:

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## Streams

- Dispersion of suspended sediments in streams.
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## More than randomness

- Can generalize to Fractional Random Walks ${ }^{[6,7,5]}$

Levy flights, Fractional Brownian Motion

- See Montroll and Shlesinger for example: "On 1/f noise and other distributions with long tails." Proc. Natl. Acad. Sci., 1982.
- In 1-d, standard deviation $\sigma$ scales as


## Random Walks

## The First Return Problem



- Extensive memory of path now matters


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## Random Walks



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$\alpha=1 / 2$ - diffusive
$\alpha>1 / 2$ - superdiffusive
$\alpha<1 / 2$ - subdiffusive

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## Outline

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## Variable transformation Basics

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## Variable Transformation

## Understand power laws as arising from

1. Elementary distributions (e.g., exponentials).
2. Variables connected by power relationships.

Random Walks
The First Return Problem

- Random variable $X$ with known distribution $P_{X}$
- Second random variable $Y$ with $y=f(x)$.
- $P_{y}(y) \mathrm{d} y=P_{x}(x) \mathrm{d} x$

- Often easier to do by hand.



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Understand power laws as arising from
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－Often easier to do by hand．
$\left|\begin{array}{c}0 \\ 0\end{array}\right|$

## Variable Transformation

Understand power laws as arising from

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$\left\lvert\, \begin{aligned} & 0 \\ & 0\end{aligned}\right.$


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1．Elementary distributions（e．g．，exponentials）．
2．Variables connected by power relationships．
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－Second random variable $Y$ with $y=f(x)$ ．
－$P_{y}(y) \mathrm{d} y=P_{x}(x) \mathrm{d} x$
$\sum_{y \mid f(x)=y} P_{x}\left(f^{-1}(y)\right) \frac{d y}{\left|f^{\prime}(f-1(y))\right|}$
－Often easier to do by
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$\left\lvert\, \begin{gathered}0 \\ 6 \\ 0\end{gathered}\right.$


## General Example

## Power-Law

Mechanisms I

- Assume relationship between $x$ and $y$ is 1-1.
- Power-law relationship between variables:


## $y=c x^{-\alpha}, \alpha>0$ <br> - Look at $y$ large and $x$ small

$$
\mathrm{d} y=\mathrm{d}\left(c x^{-\alpha}\right)
$$

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- Assume relationship between $x$ and $y$ is 1-1.
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$$
\begin{gathered}
\mathrm{d} y=\mathrm{d}\left(c x^{-\alpha}\right) \\
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$$

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& \qquad \mathrm{d} y=\mathrm{d}\left(c x^{-\alpha}\right) \\
& =c(-\alpha) x^{-\alpha-1} \mathrm{~d} x \\
& \text { invert: } \mathrm{d} x=\frac{-1}{c \alpha} x^{\alpha+1} \mathrm{~d} y
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\mathrm{~d} x=\frac{-1}{c \alpha}\left(\frac{y}{c}\right)^{-(\alpha+1) / \alpha} \mathrm{d} y
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$$

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\mathrm{~d} x=\frac{-c^{1 / \alpha}}{\alpha} y^{-1-1 / \alpha} \mathrm{d} y
\end{gathered}
$$



## Now make transformation:

$$
P_{y}(y) \mathrm{d} y=P_{x}(x) \mathrm{d} x
$$

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- If $P_{x}(x) \rightarrow$ non-zero constant as $x \rightarrow 0$ then

- If $P_{x}(x) \rightarrow x^{\beta}$ as $x \rightarrow 0$ then





## Now make transformation：

$$
\begin{gathered}
P_{y}(y) \mathrm{d} y=P_{x}(x) \mathrm{d} x \\
P_{y}(y) \mathrm{d} y=P_{x} \overbrace{\left(\left(\frac{y}{c}\right)^{-1 / \alpha}\right)}^{(x)} \overbrace{\frac{c^{1 / \alpha}}{\alpha} y^{-1-1 / \alpha} \mathrm{d} y}^{\mathrm{d} x}
\end{gathered}
$$

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－If $P_{x}(x) \rightarrow$ non－zero constant as $x \rightarrow 0$ then

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P_{y}(y) \propto y^{-1-1 / \alpha} \text { as } y \rightarrow \infty
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$$
P_{y}(y) \propto y^{-1-1 / \alpha-\beta / \alpha} \text { as } y \rightarrow \infty
$$



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## Example

## Exponential distribution

Given $P_{x}(x)=\frac{1}{\lambda} e^{-x / \lambda}$ and $y=c x^{-\alpha}$, then

$$
P(y) \propto y^{-1-1 / \alpha}+O\left(y^{-1-2 / \alpha}\right)
$$

- Exponentials arise from randomness (easy).
- More later when we cover robustness.

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## Example

## Exponential distribution

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## Outline

# Variable transformation 

## Holtsmark's Distribution



## Gravity

－Select a random point in the universe $\vec{x}$
－Measure the force of gravity $F(\vec{x})$
－Observe that $P_{F}(F) \sim F^{-5 / 2}$


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## Gravity

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Matter is concentrated in stars: ${ }^{[9]}$

- $F$ is distributed unevenly

Probability of being a distance $r$ from a single star at $\vec{x}=\overrightarrow{0}$ :


- Assume stars are distributed randomly in space

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References (oops?)

- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

- invert:
- Also invert:
$\mathrm{d} F \propto \mathrm{~d}\left(r^{-2}\right) \propto r^{-3} \mathrm{~d} r \rightarrow \mathrm{~d} r \propto r^{3} \mathrm{~d} F \propto F^{-3 / 2} \mathrm{~d} F$

Matter is concentrated in stars: ${ }^{[9]}$

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$\left|\begin{array}{c}0 \\ 0\end{array}\right|$

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## Transformation:

Using $r \propto F^{-1 / 2}, \mathrm{~d} r \propto F^{-3 / 2} \mathrm{~d} F$, and $P_{r}(r) \propto r^{2}$


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## Transformation：

Using $r \propto F^{-1 / 2}, \mathrm{~d} r \propto F^{-3 / 2} \mathrm{~d} F$ ，and $P_{r}(r) \propto r^{2}$

$$
P_{F}(F) \mathrm{d} F=P_{r}(r) \mathrm{d} r
$$

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$$
\begin{aligned}
& P_{F}(F) \mathrm{d} F=P_{r}(r) \mathrm{d} r \\
& \propto P_{r}\left(F^{-1 / 2}\right) F^{-3 / 2} \mathrm{~d} F
\end{aligned}
$$

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& \propto\left(F^{-1 / 2}\right)^{2} F^{-3 / 2} \mathrm{~d} F
\end{aligned}
$$

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& \quad=F^{-1-3 / 2} \mathrm{~d} F
\end{aligned}
$$

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\begin{gathered}
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\propto P_{r}\left(F^{-1 / 2}\right) F^{-3 / 2} \mathrm{~d} F \\
\propto\left(F^{-1 / 2}\right)^{2} F^{-3 / 2} \mathrm{~d} F \\
=F^{-1-3 / 2} \mathrm{~d} F \\
=F^{-5 / 2} \mathrm{~d} F
\end{gathered}
$$

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## Gravity：

$$
P_{F}(F)=F^{-5 / 2} \mathrm{~d} F
$$

Random Walks
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－Mean is finite．
－Variance $=\infty$ ．
－A wild distribution．
－Upshot：Random sampling of space usually safe but can end badly．．．


## Gravity：

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\begin{gathered}
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\end{gathered}
$$

## Random Walks

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## References

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## A wild distribution． <br> Upshot：Random sampling of space usually safe but can end badly．



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Random Walks
The First Return Problem Examples

Variable
transformation
Basics
Holtsmark＇s Distribution PLIPLO

## References

－Mean is finite．
－Variance $=\infty$ ．
－A wild distribution．
Upshot：Random sampling of space usually safe but can end badly．


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## Gravity:

$$
\begin{gathered}
P_{F}(F)=F^{-5 / 2} \mathrm{~d} F \\
\gamma=5 / 2
\end{gathered}
$$

- Mean is finite.
- Variance $=\infty$.
- A wild distribution.
- Upshot: Random sampling of space usually safe but can end badly...


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## Outline

## Variable transformation

## PLIPLO

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## Extreme Caution！

－PLIPLO＝Power law in，power law out
－Explain a power law as resulting from another unexplained power law．
－Yet another homunculus al
－We need mechanisms！

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－Don＇t do this！！！（slap，slap）
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- PLIPLO = Power law in, power law out
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## References I

［2］P．S．Dodds and D．H．Rothman． Scaling，universality，and geomorphology． Annu．Rev．Earth Planet．Sci．，28：571－610， 2000. pdf（ $⿴ 囗 十$ ）
［3］W．Feller．
An Introduction to Probability Theory and Its Applications，volume I．
John Wiley \＆Sons，New York，third edition， 1968.


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## References II

［4］J．T．Hack．
Studies of longitudinal stream profiles in Virginia and Maryland．
United States Geological Survey Professional Paper， 294－B：45－97，1957．pdf（ $\boxplus$ ）

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［5］E．W．Montroll and M．F．Shlesinger． On the wonderful world of random walks，volume XI of Studies in statistical mechanics，chapter 1，pages 1－121． New－Holland，New York， 1984.
［6］E．W．Montroll and M．W．Shlesinger． On $1 / f$ noise aned other distributions with long tails． Proc．Natl．Acad．Sci．，79：3380－3383，1982．pdf（ $\boxplus$ ）

## References III

［7］E．W．Montroll and M．W．Shlesinger．
Maximum entropy formalism，fractals，scaling phenomena，and $1 / f$ noise：a tale of tails．
J．Stat．Phys．，32：209－230， 1983.
［8］A．E．Scheidegger．
The algebra of stream－order numbers．
United States Geological Survey Professional Paper， 525－B：B187－B189，1967．pdf（ $⿴ 囗 十$ ）
［9］D．Sornette．
Critical Phenomena in Natural Sciences．
Springer－Verlag，Berlin，1st edition， 2003.
［10］J．S．Weitz and H．B．Fraser．
Explaining mortality rate plateaus．
Proc．Natl．Acad．Sci．，98：15383－15386， 2001.
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