Lognormals and friends

Principles of Complex Systems
CSYS/MATH 300, Spring, 2013 | #SpringPoCS2013

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Outline

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Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Alternative distributions

There are other 'heavy-tailed' distributions:

1. The Log-normal distribution (⊞)

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions (⊞)

$$P(x)\mathrm{d}x = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}}\mathrm{d}x$$

   CCDF = stretched exponential (⊞).

3. Gamma distributions (⊞), and more.

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lognormals

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- $\ln x$ is distributed according to a normal distribution with mean μ and variance σ.
- Appears in economics and biology where growth increments are distributed normally.

lognormals

- Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu + \frac{1}{2}\sigma^2}, \qquad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu-\sigma^2}.$$

- All moments of lognormals are finite.

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- All moments of lognormals are finite.

Derivation from a normal distribution

Take Y as distributed normally:

- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{d}y\, \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

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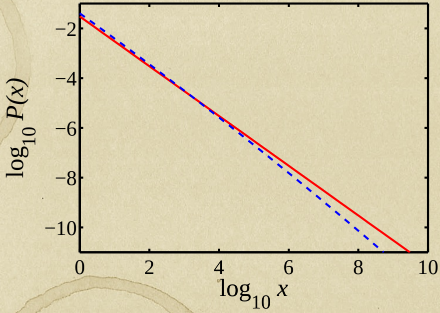
Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude!

- For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- For power law (red), $\gamma = 1$ and $c = 0.03$.

Confusion

What's happening:

- $$\ln P(x) = \ln\left\{\frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\right\}$$
- $$= -\ln x - \ln\sqrt{2\pi} - \frac{(\ln x-\mu)^2}{2\sigma^2}$$
- $$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2}-1\right)\ln x - \ln\sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$
- $\Rightarrow$ If $\sigma^2 \gg 1$ and μ,
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- This happens when (roughly)
- $$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2}-1\right)\ln x$$
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Outline

Generating lognormals:

Random multiplicative growth:

- $$x_{n+1} = r x_n$$

  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:

  $$\ln x_{n+1} = \ln r + \ln x_n$$
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- But Robert Axtell[1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
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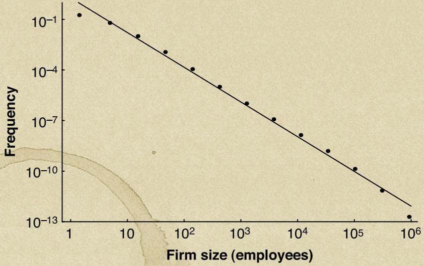

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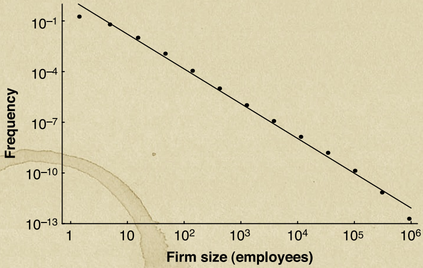

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An explanation

- Axtel (mis?)cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 2$
- The set up: N entities with size $x_i(t)$
- Generally:

$$x_i(t+1) = r x_i(t)$$

where r is drawn from some happy distribution

- Same as for lognormal but one extra piece.
- Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(r x_i(t), c \langle x_i \rangle)$$

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- where γ is implicitly given by

  $$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma - 1} - 1}{(c/N)^{\gamma - 1} - (c/N)} \right]$$

  N = total number of firms.
- $$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$$
- $$\text{Which gives} \quad \gamma \sim 1 + \frac{1}{1 - c}$$
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- $$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma-2)}{(\gamma-1)}\left[\frac{-1}{-(c/N)}\right]$$
- $$\text{Which gives} \quad \gamma \sim 1 + \frac{1}{1-c}$$
- Groovy... c small $\Rightarrow \gamma \simeq 2$

Outline

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.

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- Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$$

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- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } x/m > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
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- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take-home message: Be careful out there...

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