## The Amusing Law of Benford

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## Prof. Peter Dodds @peterdodds

Department of Mathematics \& Statistics | Center for Complex Systems | Vermont Advanced Computing Center | University of Vermont


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## Outline

## Benford＇s Law

## References



## The law of first digits

Benford's Law: ( $\boxplus$ )

$$
P(\text { first digit }=d) \propto \log _{b}\left(1+\frac{1}{d}\right)
$$

for certain sets of 'naturally' occurring numbers in base $b$
Around $30.1 \%$ of first digits are ' 1 ', compared to only $4.6 \%$ for ' 9 '

- First observed by Simon Newcornb ${ }^{14}$ in 1881 "Note on the Frequency of Use of the Different Digits in Natural Numbers'
- Independently discovered in 1938 by
- Newcomb almost always noted but Benford gets the stamp.


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## Benford＇s Law－The Law of First Digits

Observed for
－Fundamental constants（electron mass，charge，etc．）
－Utility bills
－Numbers on tax returns（ha！）
－Death rates
－Street addresses
－Numbers in newspapers
$\left|\begin{array}{l}0 \\ 0\end{array}\right|$

## Benford's Law-The Law of First Digits

Observed for

- Fundamental constants (electron mass, charge, etc.)
- Utility bills
- Numbers on tax returns (ha!)
- Death rates
- Street addresses
- Numbers in newspapers
- Cited as evidence of fraud ( $\boxplus$ ) in the 2009 Iranian elections.



## Benford's Law

## Real data:

predicted frequencies

| 30.1 | 17.6 | 12.5 | 9.7 | 7.9 | 6.7 | 5.8 | 5.1 | 4.6 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |



From 'The First-Digit Phenomenon' by T. P. Hill (1998) ${ }^{[1]}$


## Benford's Law

## Physical constants of the universe:



Taken from here $(\boxplus)$.

## Benford＇s Law

Population of countries：


Taken from here $(\boxplus)$ ．


## Essential story

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Benford's Law
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## Essential story

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\begin{aligned}
& \propto \log _{b}\left(\frac{d+1}{d}\right) \\
& \propto \log _{b}(d+1)-\log _{b}(d)
\end{aligned}
$$

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- Observe this distribution if numbers are distributed uniformly in log-space:

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P(\ln x) \mathrm{d}(\ln x) \propto 1 \cdot \mathrm{~d}(\ln x)
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- Power law distributions at work again...
- Extreme case of $\gamma \simeq 1$.


## Benford's law



## Benford's Law



Taken from here $(\boxplus)$.


## References I

[1] T. P. Hill.
The first-digit phenomenon. American Scientist, 86:358-, 1998.
[2] S. Newcomb.
Note on the frequency of use of the different digits in natural numbers. American Journal of Mathematics, 4:39-40, 1881. pdf ( $\boxplus$ )


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