Principles of Complex Systems, CSYS/MATH 300
University of Vermont, Spring 2013
Assignment 8 • code name: Ex-ter-min-ate! (⊞)

Dispersed: Thursday, April 4, 2013.
Due: By start of lecture, 11:30 am, Thursday, April 11, 2013.
Some useful reminders:
Instructor: Peter Dodds
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Office hours: 1:00 pm to 4:00 pm, Wednesday
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2013-01UVM-300

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related TeX variant).

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1. (3 + 3)

   Using Gleeson and Calahane's iterative equations below, derive the contagion condition for a vanishing seed by taking the limit $\phi_0 \to 0$ and $t \to \infty$. In lectures, we derived the discrete evolution equations for the fraction of infected nodes ϕ_t and the fraction of infected edges θ_t as follows:

   $$\phi_{t+1} = \phi_0 + (1-\phi_0)\sum_{k=0}^{\infty} P_k \sum_{j=0}^{k} \binom{k}{j} \theta_t^j (1-\theta_t)^{k-j} B_{kj},$$

   $$\theta_{t+1} = G(\theta_t;\phi_0) = \phi_0 + (1-\phi_0)\sum_{k=1}^{\infty} \frac{kP_k}{\langle k \rangle} \sum_{j=0}^{k-1} \binom{k-1}{j} {\theta_t}^j (1-\theta_t)^{k-1-j} B_{kj},$$

   where $\theta_0 = \phi_0$, and B_{kj} is the probability that a degree k node becomes active when j of its neighbors are active.

   Recall that by contagion condition, we mean the requirements of a random network for macroscopic spreading to occur.

   To connect the paper's model and notation to those of our lectures, given a specific response function F and a threshold model, the B_{kj} are given by $B_{kj} = F(j/k)$.

   Allow B_{k0} to be arbitrary (i.e., not necessarily 0 as for simple threshold functions).

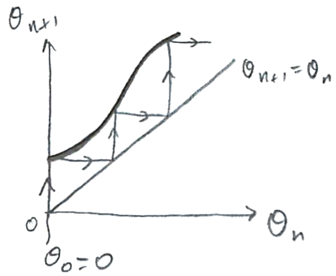
   We really only need to understand how θ_t behaves. Write the corresponding equation as $\theta_{t+1} = G(\theta_t;\phi_0)$ and determine when

(a) $G(0;\phi_0) > 0$ (spreading is for free).

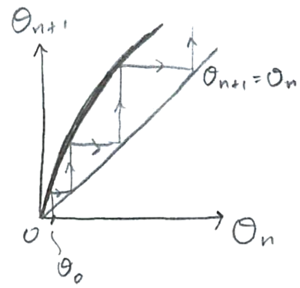
(b) $G(0;\phi_0) = 0$ and $G'(0;\phi_0) > 1$ meaning $\phi = 0$ is a unstable fixed point.

Here's a graphical hint for the three cases you need to consider as $\theta_0 \to 0$:

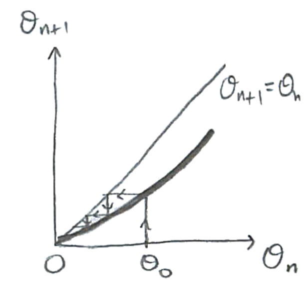
Success:



Sucesss:



Fail:



2. (3 + 3 + 3) *More on the power law stuff:*

   Take x to be the wealth held by an individual in a population of n people, and the number of individuals with wealth between x and $x + \mathrm{d}x$ to be approximately $N(x)\mathrm{d}x$.

   Given a power-law size frequency distribution $N(x) = cx^{-\gamma}$ where $x_{\mathrm{min}} \ll x \ll \infty$, determine the value of γ for which the so-called 80/20 rule holds.

   In other words, find γ for which the bottom 4/5 of the population holds 1/5 of the overall wealth, and the top 1/5 holds the remaining 4/5.

   Assume the mean is finite, i.e., $\gamma > 2$.

   (a) First determine the total wealth W in the system given $\int_{x_{\mathrm{min}}}^{\infty} \mathrm{d}x N(x) = n$.

   (b) Find γ for the 80/20 requirement.

   (c) For the γ you find, determine how much wealth $100q$ percent of the population possesses as a function of q and plot the result.

3. (3 + 3)

   (a) Let's generalize the preceding question so that $100q$ percent of the population holds $100(1-r)$ percent of the wealth.

   Show γ depends on p and q as

$$\gamma = 1 + \frac{\ln \frac{1}{(1-q)}}{\ln \frac{1}{(1-q)} - \ln \frac{1}{r}}.$$

   (Check this agrees with your result for the previous question by setting $q = 4/5$ and $r = 1/5$.

   (b) Is every pairing of q and r possible?