Principles of Complex Systems, CSYS/MATH 300
University of Vermont, Spring 2013
Assignment 7 • code name: Bacon (⊞)

Dispersed: Friday, March 22, 2013.
Due: By start of lecture, 11:30 am, Thursday, March 28, 2013.
Some useful reminders:
Instructor: Peter Dodds
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Office hours: 1:00 pm to 4:00 pm, Wednesday
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2013-01UVM-300

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related TeX variant).

Re Bacon: Rhyming slang doesn't always rhyme (hemiteleia). And then there's the Six Degrees thing. Everything is connected.

1. (3 + 3 + 3)

   Solve Krapivsky-Redner's model for the pure linear attachment kernel $A_k = k$.

   Starting point:
   $$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$
   with $n_0 = 0$.

   (a) Determine n_1.

   (b) Find a recursion relation for n_k in terms of n_{k-1}.

   (c) Now find
   $$n_k = \frac{4}{k(k+1)(k+2)}$$
   for all k and hence determine γ.

2. (3 + 3 + 3)

   From lectures, complete the analysis for the Krapivsky-Redner model with attachment kernel:
   $$A_1 = \alpha \text{ and } A_k = k \text{ for } k \geq 2.$$

Find the scaling exponent $\gamma = \mu + 1$ by finding μ. From lectures, we assumed a linear growth in the sum of the attachment kernel weights $\mu t = \sum_{k=1}^{\infty} N_k(t) A_k$, with $\mu = 2$ for the standard kernel $A_k = k$.

We arrived at this expression for μ which you can use as your starting point:

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

(a) Show that the above expression leads to

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

Hint: you'll want to separate out the $j = 1$ case for which $A_j = \alpha$.

(b) Now use result that [2]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

to find the connection

$$\mu(\mu - 1) = 2\alpha,$$

and show this leads to

$$\mu = \frac{1 + \sqrt{1 + 8\alpha}}{2}.$$

(c) Interpret how varying α affects the exponent γ, explaining why $\alpha < 1$ and $\alpha > 1$ lead to the particular values of γ that they do.

3. (3 + 3)

Determine the clustering coefficient for toy model small-world networks [3] as a function of the rewiring probability p. Find C_1, the average local clustering coefficient:

$$C_1(p) = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i = \frac{1}{N} \sum_{i=1}^{N} \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2}$$

where N is the number of nodes, $a_{ij} = 1$ if nodes i and j are connected, and $\mathcal{N}_i$ indicates the neighborhood of i.

As per the original model, assume a ring network with each node connected to a fixed, even number m local neighbors ($m/2$ on each side). Take the number of nodes to be $N \gg m$.

Start by finding $C_1(0)$ and argue for a $(1-p)^3$ correction factor to find an approximation of $C_1(p)$.

Hint 1: you can think of finding C_1 as averaging over the possibilities for a single node.

Hint 2: assume that the degree of individual nodes does not change with rewiring but rather stays fixed at m. In other words, take the average degree of individuals as the degree of a randomly selected individual.

For what value of p is $C_1 \simeq 1/2$?

(3 points for set up, 3 for solving.)

4. Optional (3 + 3):

   From lectures:

   (a) Starting from the recursion relation

   $$n_k = \frac{A_{k-1}}{\mu + A_k} n_{k-1},$$

   and $n_1 = \mu/(\mu + A_1)$, show that the expression for n_k for the Krapivsky-Redner model with an asymptotically linear attachment kernel A_k is:

   $$\frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}.$$

   (b) Now, show that if $A_k \to k$, we obtain $n_k \to k^{-\mu-1}$.

5. Optional (3 + 3):

   Consider a modified version of the Barabàsi-Albert (BA) model [1] where two possible mechanisms are now in play. As in the original model, start with m_0 nodes at time $t = 0$. Let's make these initial guys connected such that each has degree 1. The two mechanisms are:

   M1: With probability p, a new node of degree 1 is added to the network. At time $t+1$, a node connects to an existing node j with probability

   $$P(\text{connect to node } j) = \frac{k_j}{\sum_{i=1}^{N(t)} k_i} \tag{1}$$

   where k_j is the degree of node j and $N(t)$ is the number of nodes in the system at time t.

M2: With probability $q = 1 - p$, a randomly chosen node adds a new edge, connecting to node j with the same preferential attachment probability as above.

Note that in the limit $q = 0$, we retrieve the original BA model (with the difference that we are adding one link at a time rather than m here).

In the long time limit $t \to \infty$, what is the expected form of the degree distribution P_k?

Do we move out of the original model's universality class?

(3 points for set up, 3 for solving.)

References

[1] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. Science, 286:509–511, 1999.

[2] P. L. Krapivsky and S. Redner. Organization of growing random networks. Phys. Rev. E, 63:066123, 2001.

[3] D. J. Watts and S. J. Strogatz. Collective dynamics of 'small-world' networks. Nature, 393:440–442, 1998.