Principles of Complex Systems, CSYS/MATH 300
University of Vermont, Spring 2013
Assignment 6 • code name: Dogmatix (⊞)

Dispersed: Thursday, February 21, 2013.
Due: By start of lecture, 11:30 am, Thursday, March 21, 2013.
Some useful reminders:
Instructor: Peter Dodds
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Office hours: 1:00 pm to 4:00 pm, Wednesday
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2013-01UVM-300

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related TeX variant).

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1. (3 + 3) (Optional) The 1-d percolation problem:

   Consider an infinite 1-d lattice forest with a tree present at any site with probability p.

   - Find the distribution of forest sizes as a function of p. Do this by moving along the 1-d world and figuring out the probability that any forest you enter will extend for a total length ℓ.
   - Find p_c, the critical probability for which a giant component exists.

2. (3 + 3 + 3)

   A courageous coding festival:

   Code up the discrete HOT model in 1-d. Let's see if we find any of these super-duper power laws everyone keeps talking about. We'll follow the same approach as the 2-d forest discussed in lectures.

   Main goal: extract yield curves as a function of the design D parameter as described below.

   Suggested simulations elements:

   - $N = 10^4$ as a start. Then see if $N = 10^5$ or $N = 10^6$ is possible.
   - Start with no trees.

- Probability of a spark at the ith site: $P(i) \propto e^{-i/\ell}$ where i is tree position ($i = 1$ to N). (You will need to normalize this properly.) The quantity ℓ is the characteristic scale for this distribution; try $\ell = 2 \times 10^5$.
- Consider a design problem of $D = 1$, 2, $N^{1/2}$, and N. (If $N^{1/2}$ and N are too much, you can drop them. Perhaps sneak out to $D = 3$.) Recall that the design problem is to test D randomly chosen placements of the next tree against the spark distribution.
- For each test tree, measure the average yield (number of trees left) with $n = 100$ randomly selected sparks. Select the tree location with the highest average yield and plant a tree there.
- Add trees until the linear forest is full, measuring average yield as a function of trees added.
- Only trees and adjacent trees burn. In effect, you will be burning un-treed intervals of the line (much less complicated than 2-d).

(a) Plot the yield curves for each value of D.

(b) Identify peak yield for each value of D.

(c) Plot distributions of connected tree interval sizes at peak yield (you will have to rebuild forests and stop at the peak yield value of D to find these distributions.

Hint: keeping a list of un-treed locations will make choosing the next location easier. Hopefully.

3. (Optional)

In lectures on lognormals and other heavy-tailed distributions, we came across a fun and interesting integral when considering organization size distributions arising from growth processes with variable lifespans.

Show that

$$P(x) = \int_{t=0}^{\infty} \lambda e^{-\lambda t} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) \mathrm{d}t$$

leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}},$$

and therefore two different scaling regimes. Enjoyable suffering may be involved. Really enjoyable suffering.

Hints and steps:

- Make the substitution $t = u^2$ to find an integral of the form

$$I_1(a, b) = \int_0^\infty \exp\left(-au^2 - b/u^2\right) \mathrm{d}u$$

where in our case $a = \lambda$ and $b = (\ln x/m)^2/2$.

- Substitute $au^2 = t^2$ into the above to find

$$I_1(a, b) = \frac{1}{\sqrt{a}} \int_0^\infty \exp\left(-t^2 - ab/t^2\right) \mathrm{d}t$$

- Now work on this integral:

$$I_2(r) = \int_0^\infty \exp\left(-t^2 - r/t^2\right) \mathrm{d}t$$

where $r = ab$.

- Differentiate I_2 with respect to r to create a simple differential equation for I_2 You will need to use the substitution $u = \sqrt{r}/t$ and your differential equation should be of the form

$$\frac{\mathrm{d}I_2(r)}{\mathrm{d}r} = -(\text{something}) I_2(r).$$

- Solve the differential equation you find. To find the constant of integration, you can evaluate $I_2(0)$ separately:

$$I_2(0) = \int_0^\infty \exp(-t^2) \mathrm{d}t,$$

where our friend $\Gamma(1/2)$ comes into play.

4. (3 + 3 + 3 + 3) This question is all about pure finite and infinite random networks

   We'll define a finite random network as follows. Take N labelled nodes and add links between each pair of nodes with probability p.

   (a) i. For a random node i, determine the probability distribution for its number of friends k, $P_k(p, N)$.

   ii. What kind of distribution is this?

   iii. What does this distribution tend toward in the limit of large N, if p is fixed?

   (b) Using $P_k(p, N)$, determine the average degree. Does your answer seem right intuitively?

(c) Show that in the limit of $N \to \infty$ but with mean held constant, we obtain a Poisson degree distribution.

Hint: to keep the mean constant, you will need to change p.

(d)
   i. Compute the clustering coefficients C_1 and C_2 for standard random networks.
   ii. Explain how your answers make sense.
   iii. What happens in the limit of an infinite random network with finite mean?