Mechanisms for Generating Power-Law Size Distributions I

Principles of Complex Systems
CSYS/MATH 300, Fall, 2011

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Outline

Random Walks
The First Return Problem
Examples

Variable transformation
Basics
Holtsmark's Distribution
PLIPLO

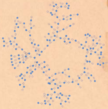
References

Mechanisms

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

A powerful story in the rise of complexity:

- structure arises out of randomness.



Mechanisms

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

A powerful story in the rise of complexity:

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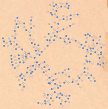
- Exhibit A: Random walks... (⊞)



Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

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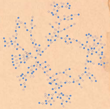

Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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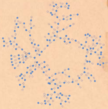
Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$



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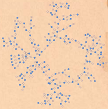
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Variances sum: (⊞)*

$$\text{Var}(x_t) = \text{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \text{Var}(\epsilon_i) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.

Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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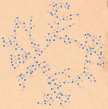
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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$$\sigma = t^{1/2}$$

$\Rightarrow$ A non-trivial power-law arises out of additive aggregation or accumulation.

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Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Random walks are weirder than you might think...

For example:

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is...

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Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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See Feller, [2] Intro to Probability Theory, Volume I

Random walks

In fact:

$$\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$$

Even crazier:

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

In fact:

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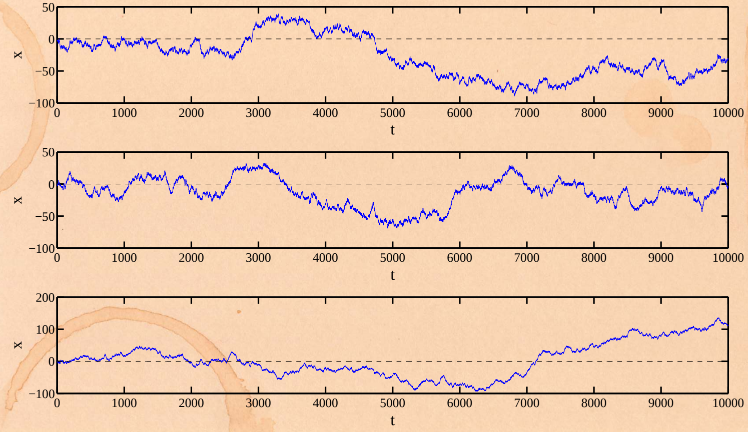
Even crazier:

The expected time between tied scores = ∞!

Random walks—some examples



Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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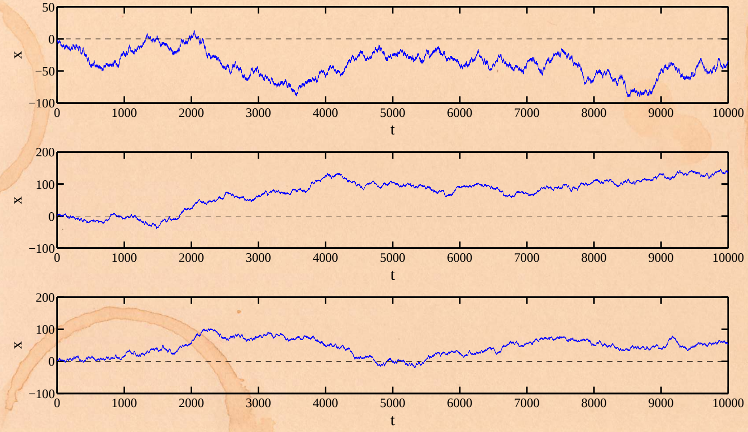

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent
2. Some physical structures may result from random walks
3. We'll start to see how different scalings relate to each other

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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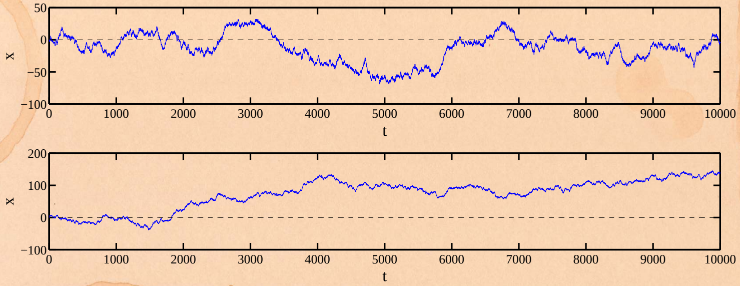
Outline

Random Walks
The First Return Problem
Examples

Variable transformation
Basics
Holtsmark's Distribution
PLIPLO

References

Random Walks



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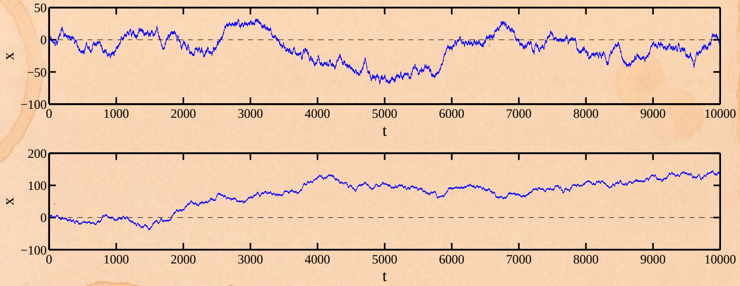
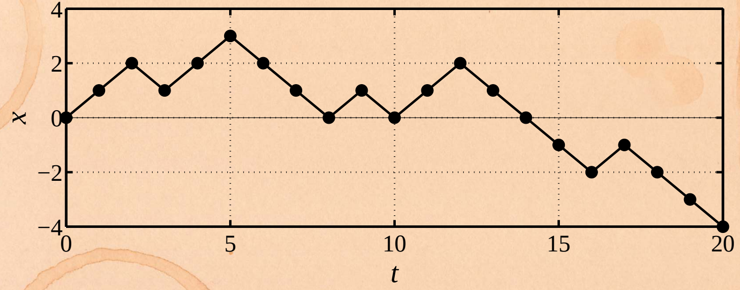


Again: expected time between ties = ∞...
Let's find out why... [2]

First Returns



First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

For random walks in 1-d:

- Return can only happen when $t = 2n$.

First Returns

Random Walks
The First Return Problem
Examples
Variable
transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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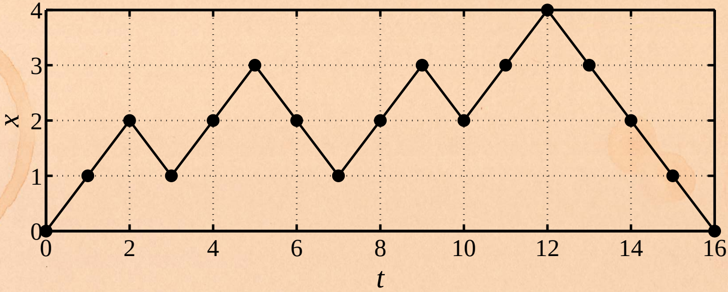
- The problem

$$P_{\text{fr}}(2n) = 2Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_{2n} = 0)$$

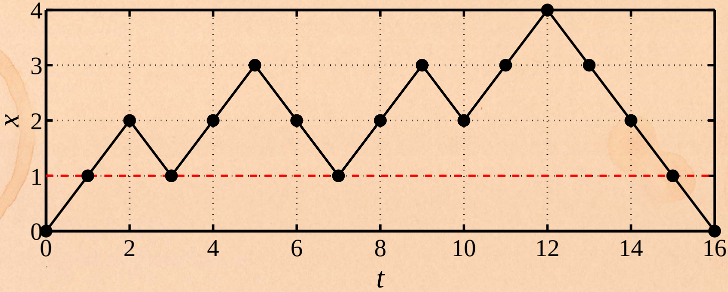
First Returns



- A useful restatement: $P_{fr}(2n) = 2 \cdot \frac{1}{2} Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$
- Want walks that can return many times to $x = 1$.
- (The $\frac{1}{2}$ accounts for stepping to 2 instead of 0 at $t = 2n$.)

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

First Returns



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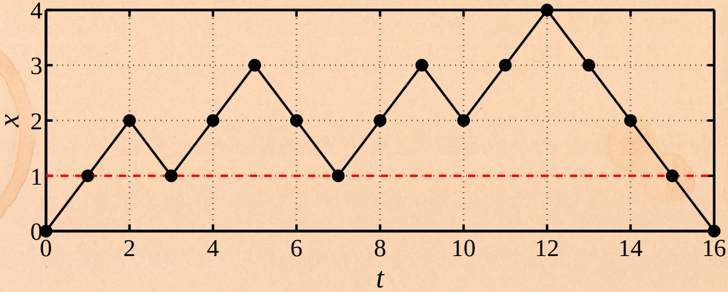

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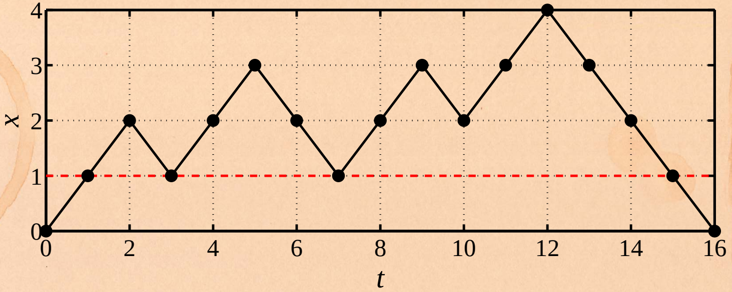
First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



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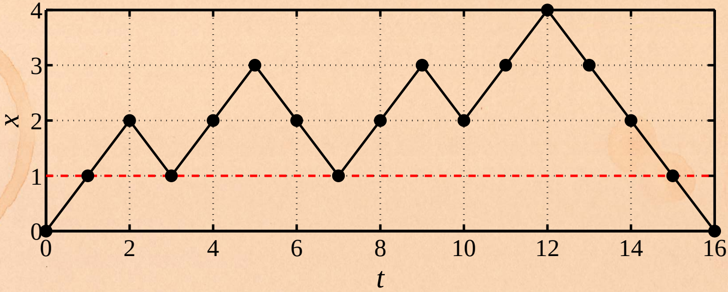

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

First Returns

- Counting problem (combinatorics/statistical mechanics)
- Use a method of images
- Define $N(i, j, t)$ as the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Subtract how many hit $x = 0$.

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Key observation:
of t-step paths starting and ending at $x = 1$ and hitting $x = 0$ at least once

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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= # of t-step paths starting at $x = -1$ and ending at $x = 1$

First Returns

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First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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= # of t-step paths starting at $x = -1$ and ending at $x = 1$
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So $N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$

First Returns

Random Walks
The First Return Problem
Examples
Variable
transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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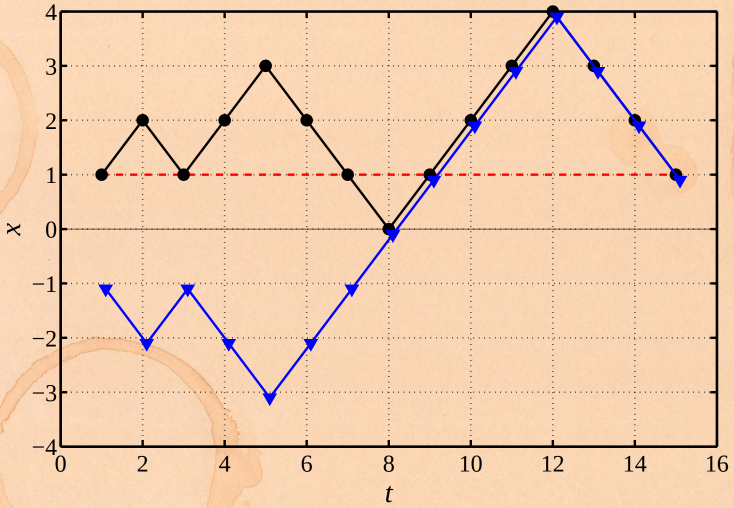
See this 1-1 correspondence visually...

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

- For any path starting at $x = 1$ that hits 0, there is a unique matching path starting at $x = -1$.
- Matching path first mirrors and then tracks.

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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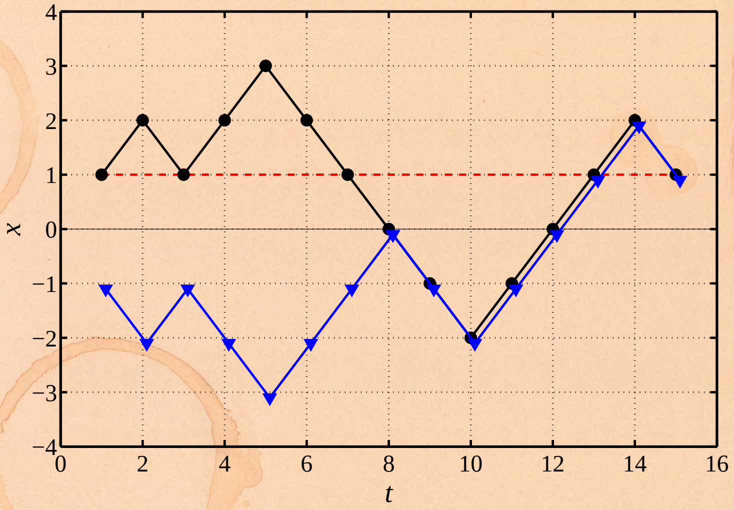
First Returns



Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

- Next problem: what is $N(i, j, t)$?
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First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

We now have

$$N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$$

where

$$N(i, j, t) = \binom{t}{(t+j-i)/2}$$

First Returns

Insert question from assignment 4 (⊞)

Find $N_{\text{first return}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$.

- Normalized Number of Paths gives Probability
- Total number of possible paths $= 2^{2n}$
-

$$P_{\text{first return}}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$$

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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- Same scaling holds for continuous space/time walks.
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$$P(t) \propto t^{-3/2},\ \gamma = 3/2$$

- $P(t)$ is normalizable
- Recurrence: Random walker always returns to origin
- Moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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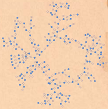
First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions
- For $d = 1$, $\gamma = 3/2 \rightarrow \langle t \rangle = \infty$
- Even though walker must return, expect a long wait...



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First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

On finite spaces:

- In any finite volume, a random walker will visit every site with equal probability
- Random walking $\equiv$ Diffusion
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

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Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random walks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random walks on

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree
- Equal probability still present:
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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Outline

Random Walks

The First Return Problem

Examples

Variable transformation

Basics

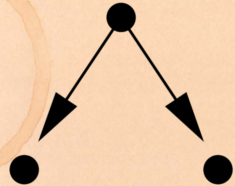
Holtsmark's Distribution

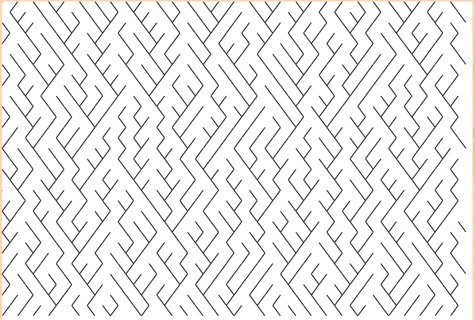
PLIPLO

References

Scheidegger Networks [4, 1]

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References





- Triangular lattice
- 'Flow' is southeast or southwest with equal probability.

Scheidegger Networks

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

- Creates basins with random walk boundaries
- Observe Subtracting one random walk from another gives random walk with increments

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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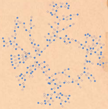
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



Connections between Exponents

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
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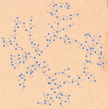

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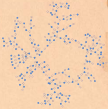
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



Connections between Exponents

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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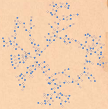

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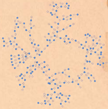
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



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Connections between Exponents

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

- Both basin area and length obey power law distributions
- Observed for real river networks
- Typically: $1.3 < \tau < 1.5$ and $1.5 < \gamma < 2$
- Smaller basins more allometric ($h > 1/2$)
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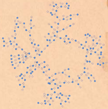

Connections between Exponents

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Connections between Exponents

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Connections between Exponents

- Generalize relationship between area and length
- Hack's law [3]:

$$\ell \propto a^h$$

where $0.5 \lesssim h \lesssim 0.7$

- Redo calc with γ, τ, and h.

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

- Generalize relationship between area and length
- Hack's law [3]:

$$\ell \propto a^h$$

  where $0.5 \lesssim h \lesssim 0.7$
- Redo calc with γ, τ, and h.

Connections between Exponents

- Given

$$\ell \propto a^h, \; P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = ha^{h-1}\mathrm{d}a$
- $Pr(\text{basin area} = a)\mathrm{d}a$

$$= Pr(\text{basin length} = \ell)\mathrm{d}\ell$$
$$\propto \ell^{-\gamma}\mathrm{d}\ell$$
$$\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$$
$$= a^{-(1+h(\gamma-1))}\mathrm{d}a$$

- $$\boxed{\tau = 1 + h(\gamma - 1)}$$

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Connections between Exponents

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies:

$$\tau = 2 - h$$

$$\gamma = 1/h$$

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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- Expect scaling relations where power laws are found

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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- Simplify system description
- Expect scaling relations where power laws are found
- Characterize universality class with independent exponents

Other First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Failure

- A very simple model of failure/death:
- x_t = entity's 'health' at time t
- x_0 could be > 0.
- Entity fails when x hits 0.

Streams

- Dispersion of suspended sediments in streams.
- Long times for clearing.

Other First Returns

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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More than randomness

- Can generalize to Fractional Random Walks
- Levy flights, Fractional Brownian Motion
- In 1-d,

$$\sigma \sim t^{\alpha}$$

- Extensive memory of path now matters...

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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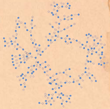
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Outline

Random Walks
The First Return Problem
Examples

Variable transformation
Basics
Holtsmark's Distribution
PLIPLO

References

Variable Transformation

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Understand power laws as arising from

1. elementary distributions (e.g., exponentials)

Variable Transformation

Understand power laws as arising from

1. elementary distributions (e.g., exponentials)
2. variables connected by power relationships

Variable Transformation

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.
- $P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$
  $= \sum_{y|f(x)=y} P_x(f^{-1}(y)) \frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$
- Often easier to do by hand...

Figure...

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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- Often easier to do by hand...

General Example

Assume relationship between x and y is 1-1.

- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
- $$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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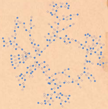
Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



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Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

- If $P_x(x) \to$ non-zero constant as $x \to 0$ then

$$P_y(y) \propto y^{-1-1/\alpha} \text{ as } y \to \infty.$$

- If $P_x(x) \to x^\beta$ as $x \to 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \to \infty.$$

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

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Exponential distribution

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- Exponentials arise from randomness...
- More later when we cover robustness.

Outline

Random Walks

The First Return Problem

Examples

Variable transformation

Basics

Holtsmark's Distribution

PLIPLO

References

Gravity

- Select a random point in the universe $\vec{x}$
- (possible all of space-time)
- Measure the force of gravity $F(\vec{x})$
- Observe that $P_F(F) \sim F^{-5/2}$.

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Ingredients [5]

Matter is concentrated in stars:

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

$$r \propto F^{-1/2}$$

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Transformation

- $$\mathrm{d}F \propto \mathrm{d}(r^{-2})$$
- $$\propto r^{-3}\mathrm{d}r$$
- invert:
  $$\mathrm{d}r \propto r^{3}\mathrm{d}F$$
- $$\propto F^{-3/2}\mathrm{d}F$$

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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- $$dF \propto d(r^{-2})$$
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- invert:
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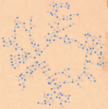

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

Transformation

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$ and $\boxed{P_r(r) \propto r^2}$

- $$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$
- $$\propto P_r(F^{-1/2})F^{-3/2}\mathrm{d}F$$
- $$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$
- $$= F^{-1-3/2}\mathrm{d}F$$
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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$ and $\boxed{P_r(r) \propto r^2}$

- $$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$
- $$\propto P_r(F^{-1/2})F^{-3/2}\mathrm{d}F$$
- $$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$
- $$= F^{-1-3/2}\mathrm{d}F$$
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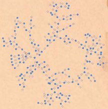
Gravity

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

- $$\gamma = 5/2$$
- Mean is finite
- Variance = ∞
- A wild distribution
- Random sampling of space usually safe but can end badly...

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References



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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Outline

Random Walks
The First Return Problem
Examples

Variable transformation
Basics
Holtsmark's Distribution
PLIPLO

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Caution!

- PLIPLO = Power law in, power law out
- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument (⊞)...
- Don't do this!!! (slap, slap)
- We need mechanisms!

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

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Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
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References I

[1] P. S. Dodds and D. H. Rothman.
Scaling, universality, and geomorphology.
Annu. Rev. Earth Planet. Sci., 28:571–610, 2000.
pdf (⊞)

[2] W. Feller.
An Introduction to Probability Theory and Its Applications, volume I.
John Wiley & Sons, New York, third edition, 1968.

[3] J. T. Hack.
Studies of longitudinal stream profiles in Virginia and Maryland.
United States Geological Survey Professional Paper, 294-B:45–97, 1957. pdf (⊞)

References II

Random Walks
The First Return Problem
Examples
Variable transformation
Basics
Holtsmark's Distribution
PLIPLO
References

[4] A. E. Scheidegger.
The algebra of stream-order numbers.
United States Geological Survey Professional Paper, 525-B:B187–B189, 1967. pdf (⊞)

[5] D. Sornette.
Critical Phenomena in Natural Sciences.
Springer-Verlag, Berlin, 2nd edition, 2003.