# Lognormals and friends Principles of Complex Systems CSYS/MATH 300. Fall. 2011

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# Lognormals and friends

### Lognormals

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Random Multiplicative
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# There are other 'heavy-tailed' distributions:

1. The Log-normal distribution (⊞)

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions (⊞]

$$P(x)dx = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}} dx$$

 $CCDF = stretched exponential (<math>\boxplus$ ).

Gamma distributions (⊞), and more.



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3. Gamma distributions  $(\boxplus)$ , and more.



# The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- In x is distributed according to a normal distribution with mean  $\mu$  and variance  $\sigma$ .
- Appears in economics and biology where growth increments are distributed normally.



Standard form reveals the mean  $\mu$  and variance  $\sigma^2$  of the underlying normal distribution:

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For lognormals:

$$\mu_{ ext{lognormal}} = e^{\mu + rac{1}{2}\sigma^2}, \qquad ext{median}_{ ext{lognormal}} = e^{\mu},$$
  $\sigma_{ ext{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \qquad ext{mode}_{ ext{lognormal}} = e^{\mu - \sigma^2}.$ 

All moments of lognormals are finite.

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All moments of lognormals are finite.



# Derivation from a normal distribution

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# Take *Y* as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma}dy \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

# Set $Y = \ln X$ :

▶ Transform according to P(x)dx = P(y)dy

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$

$$\Rightarrow P(x)dx = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

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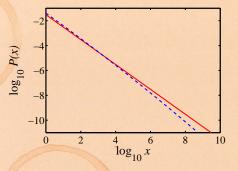
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# Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude!

- For lognormal (blue),  $\mu = 0$  and  $\sigma = 10$ .
- For power law (red),  $\gamma = 1$  and c = 0.03.

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$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right)\ln x - \ln\sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$

$$ightharpoonup$$
  $\Rightarrow$  If  $\sigma^2 \gg 1$  and  $\mu$ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

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**Empirical Confusability** Growth Model







# What's happening:

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# Confusion

Expect -1 scaling to hold until  $(\ln x)^2$  term becomes significant compared to  $(\ln x)$ .

This happens when (roughly)

 $(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1\right) \ln x$ 

 $\log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e^{-\mu}$ 

 $0.05(\sigma^2 - \mu)$ 

ou find a -1 exponent,

you may have a lognormal distribution...

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→ If you find a -1 exponent, you may have a lognormal distribution...





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# Generating lognormals:

Random multiplicative growth:

$$x_{n+1} = rx_n$$

# where r > 0 is a random growth variable

- ► (Shrinkage is allowed)
- ▶ In log space, growth is by addition:

$$\ln x_{n+1} = \ln r + \ln x_n$$

- ightharpoonup  $\Rightarrow$  In  $x_n$  is normally distributed
- $ightharpoonup \Rightarrow x_n$  is lognormally distributed

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# Lognormals or power laws?

▶ Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ( $\gamma \simeq$  1).

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- Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ( $\gamma \simeq 1$ ).
- But Robert Axtell [1] (2001) shows a power law fits the data very well with  $\gamma = 2$ , not  $\gamma = 1$  (!)

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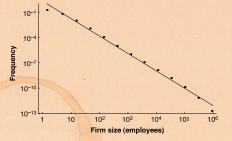
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Freq  $\propto$  (size) $^{-\gamma}$   $\gamma \simeq$  2

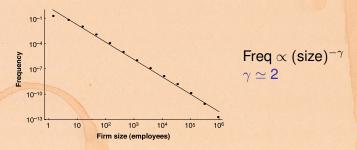
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 One mechanistic piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size. Lognormals and

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# An explanation

Axtel (mis?)cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent  $\gamma \simeq$  1

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$$x_i(t+1)=rx_i(t)$$

where r is drawn from some happy distribution

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- The set up: N entities with size  $x_i(t)$
- Generally:

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- Same as for lognormal but one extra piece.
- Each  $x_i$  cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(rx_i(t), c\langle x_i\rangle)$$

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6 (⊞)

Find 
$$P(x) \sim x^{-\gamma}$$

lacktriangle where  $\gamma$  is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[ \frac{(c/N)^{\gamma - 1} - 1}{(c/N)^{\gamma - 1} - (c/N)} \right]$$

N =total number of firms.

Now, if 
$$c/N \ll 1$$
,  $N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[ \frac{-1}{-(c/N)} \right]$ 

Which gives 
$$\gamma \sim 1 + \frac{1}{1-\alpha}$$

▶ Groovy...  $c \text{ small} \Rightarrow \gamma \simeq 2$ 

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$$lacktriangle$$
 where  $\gamma$  is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[ \frac{(c/N)^{\gamma - 1} - 1}{(c/N)^{\gamma - 1} - (c/N)} \right]$$

N = total number of firms.

Now, if 
$$c/N \ll 1$$
,  $N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[ \frac{-1}{-(c/N)} \right]$ 

Which gives 
$$\gamma \sim 1 + \frac{1}{1-c}$$

• Groovy...  $c \text{ small} \Rightarrow \gamma \simeq 2$ 

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### Outline

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# Ages of firms/people/... may not be the same

- ► Allow the number of updates for each size *x<sub>i</sub>* to vary
- ► Example:  $P(t)dt = ae^{-at}dt$  where t = age.
- ▶ Back to no bottom limit: each *x<sub>i</sub>* follows a lognormal
- ► Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that  $\sigma \sim t$  and  $\mu = \ln m$ )

▶ Now averaging different lognormal distributions.

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$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) dt$$

- ▶ Insert question from assignment 6 (⊞)
- Some enjoyable suffering leads to:

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$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$$

Depends on sign of  $\ln x/m$ , i.e., whether x/m > /m < 1.

 $P(x) \propto \begin{cases} x^{-1+\sqrt{2x}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2x}} & \text{if } x/m > 1 \end{cases}$ 

'Break' in scaling (not uncommon)

 $\mathsf{ole} ext{-}\mathsf{Pareto}$  distribution  $(\boxplus)$ 

by Montroll and Shlesinger [7, 8]

Later, Hüberman and Adamic [3, 4]. Number of pages

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# $P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$

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- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes.
   the double Pareto distribution appears
  - Take-home message. Be careful out there:

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