Principles of Complex Systems, CSYS/MATH 300
University of Vermont, Fall 2011
Assignment 6

Dispersed: Monday, November 14, 2011.
Due: By start of lecture, 11:30 am, Thursday, December 1, 2011.
Some useful reminders:
Instructor: Peter Dodds
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Office hours: 12:50 pm to 3:50 pm, Wednesday
Course website: http://www.uvm.edu/∼pdodds/teaching/courses/2011-08UVM-300

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related TeX variant).

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1. *Highly Optimized Tolerance:*

   This question is based on Carlson and Doyle's 1999 paper "Highly optimized tolerance: A mechanism for power laws in design systems."[1] In class, we made our way through a discrete version of a toy HOT model of forest fires. This paper revolves around the equivalent continuous model's derivation.

   Our interest is in Table I on p. 1415:

$p(x)$	$p_{\rm cum}(x)$	$P_{\rm cum}(A)$
$x^{-(q+1)}$	x^{-q}	$A^{-\gamma(1-1/q)}$
e^{-x}	e^{-x}	$A^{-\gamma}$
e^{-x^2}	$x^{-1}e^{-x^2}$	$A^{-\gamma}[\log(A)]^{-1/2}$

   and Equation 8 on the same page:

$$P_{\rm cum}(A) = \int_{p^{-1}(A^{-\gamma})}^{\infty} p(\mathbf{x})\mathrm{d}\mathbf{x} = p_{\rm cum}\left(p^{-1}\left(A^{-\gamma}\right)\right),$$

   where $\gamma = \alpha + 1/\beta$.

   Please note that $P_{\rm cum}(A)$ for $x^{-(q+1)}$ is not correct. Find the right one!

   Here, $A(\mathbf{x})$ is the area connected to the point $\mathbf{x}$ (think connected patch of trees for forest fires). The cost of a 'failure' (e.g., lightning) beginning at $\mathbf{x}$ scales as

$A(\mathbf{x})^\alpha$ which in turn occurs with probability $p(\mathbf{x})$. The function p^{-1} is the inverse function of p.

Resources associated with point $\mathbf{x}$ are denoted as $R(\mathbf{x})$ and area is assumed to scale with resource as $A(\mathbf{x}) \sim R^{-\beta}(\mathbf{x})$.

Finally, p_{cum} is the complementary cumulative distribution function for p.

As per the table, determine $p_{\mathrm{cum}}(x)$ and $P_{\mathrm{cum}}(A)$ for the following (3 pts each):

(a) $p(x) \sim x^{-(q+1)}$,

(b) $p(x) \sim e^{-x}$, and

(c) $p(x) \sim e^{-x^2}$.

2. (3 + 3 + 3) *More on the power law stuff:*

   Take x to be the wealth held by an individual in a population of n people, and the number of individuals with wealth between x and $x + \mathrm{d}x$ is approximately $N(x)\mathrm{d}x$.

   Given a power-law size frequency distribution $N(x) = cx^{-\gamma}$ where $a \ll x \ll \infty$, determine the value of γ for which the so-called 80/20 rule holds.

   In other words, find γ for which the bottom 4/5 of the population holds 1/5 of the overall wealth the top 1/5 holds the remaining 4/5.

   (a) First determine the total wealth W in the system given $\int_a^\infty \mathrm{d}x N(x) = n$.

   (b) Find γ for the 80/20 requirement.

   (c) For the γ you find, determine how much wealth $100q$ percent of the population possesses as a function of q and plot the result.

3. (3 + 3)

   (a) Generalize the preceding question to find γ such that $100q$ percent of the population holds $100(1-r)$ percent of the wealth.

   (b) Is every pairing of q and r possible?

4. (Optional)

   In lectures on lognormals and other heavy-tailed distributions, we came across a fun and interesting integral when considering organization size distributions arising from growth processes with variable lifespans.

   Show that

$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) \mathrm{d}t$$

   leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}},$$

   and therefore two different scaling regimes. Enjoyable suffering may be involved.

References

[1] J. M. Carlson and J. Doyle. Highly optimized tolerance: A mechanism for power laws in designed systems. *Phys. Rev. E*, 60(2):1412–1427, 1999.