Singular Value Decomposition Matrixology (Linear Algebra)—Lecture 25/25 MATH 124, Fall, 2011

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The Fundamental Theorem of Linear Algebra

Approximating matrices with SVD







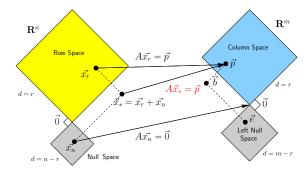
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Lecture 25/25: Singular Value Decomposition

Best solution \vec{x}_* when $\vec{b} = \vec{p} + \vec{e}$:

The Fundamental Theorem of Linea Algebra

Approximating matrices with SVD



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Approximating matrices with SVD





The Fundamental Theorem of Linear Algebra

Outline

Lecture 25/25: Singular Value Decomposition

The Fundamental Theorem of Linear Algebra

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Fundamental Theorem of Linear Algebra

Lecture 25/25: Singular Value Decomposition

Now we see:

- ► Each of the four fundamental subspaces has a 'best' orthonormal basis
- ▶ The \hat{v}_i span R^n
- ▶ We find the \hat{v}_i as eigenvectors of A^TA .
- ▶ The \hat{u}_i span R^m
- ▶ We find the \hat{u}_i as eigenvectors of AA^T .

Happy bases

- \blacktriangleright { $\hat{v}_1, \ldots, \hat{v}_r$ } span Row space
- $\{\hat{v}_{r+1}, \dots, \hat{v}_n\}$ span Null space
- $\{\hat{u}_1, \dots, \hat{u}_r\}$ span Column space
- $\{\hat{u}_{r+1}, \dots, \hat{u}_m\}$ span Left Null space





Lecture 25/25:

Singular Value Decomposition

Approximating matrices with SVD

Fundamental Theorem of Linear Algebra

- ▶ Applies to any $m \times n$ matrix A.
- ightharpoonup Symmetry of A and A^{T} .

Where \vec{x} lives:

- ▶ Row space $C(A^T) \subset R^n$.
- ▶ (Right) Nullspace $N(A) \subset R^n$.
- ▶ dim $C(A^{T})$ + dim N(A) = r + (n r) = n
- ▶ Orthogonality: $C(A^T) \otimes N(A) = R^n$

Where \vec{b} lives:

- ▶ Column space $C(A) \subset R^m$.
- ▶ Left Nullspace $N(A^T) \subset R^m$.
- ▶ dim C(A) + dim $N(A^{T}) = r + (m r) = m$
- ▶ Orthogonality: $C(A) \otimes N(A^T) = R^m$

Lecture 25/25: Singular Value

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Approximating matrices with SVD

Fundamental Theorem of Linear Algebra

How $A\vec{x}$ works:

$$A\hat{\mathbf{v}}_i = \sigma_i \hat{\mathbf{u}}_i$$
 for $i = 1, \ldots, r$.

and

$$\hat{A\hat{v}_i} = \hat{0}$$
 for $i = r + 1, \ldots, n$.

Matrix version:

$$A = U\Sigma V^{\mathrm{T}}$$

- ▶ A sends each $\hat{v}_i \in C(A^T)$ to its partner $\hat{u}_i \in C(A)$ with a positive stretch/shrink factor $\sigma_i > 0$.
- A is diagonal with respect to these bases.
- When viewed in the right way, every A is a diagonal matrix Σ.











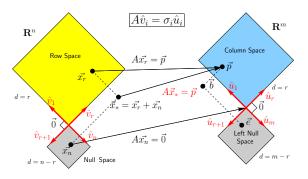
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The complete big picture:

Lecture 25/25: Singular Value Decomposition



Approximating matrices with SVD







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Image approximation (80x60)

Lecture 25/25: Singular Value Decomposition

The Fundamental Theorem of Linear Algebra

Approximating matrices with SVD

Idea: use SVD to approximate images

- ► Interpret elements of matrix *A* as color values of an image.
- ▶ Truncate series SVD representation of *A*:

$$A = U\Sigma V^{\mathrm{T}} = \sum_{i=1}^{r} \sigma_{i} \hat{u}_{i} \hat{v}_{i}^{\mathrm{T}}$$

- ▶ Use fact that $\sigma_1 \ge \sigma_2 \ge ... \ge \sigma_r > 0$.
- ▶ Rank $r = \min(m, n)$.
- ▶ Rank r = # of pixels on shortest side (usually).
- ► For color: approximate 3 matrices (RGB).





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