Positive Definite Matrices

Matrixology (Linear Algebra)—Lecture 22/25 MATH 124, Fall, 2011

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont















Positive Definite Matrices (PDMs)

Positive Definite Matrices

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What a PDM is...

Identifying PDMs

Gaussian elimination
Principle Axis Theorem

Nutsnell

Optional materia





Outline

Positive Definite Matrices

Motivation... What a PDM is... Identifying PDMs Completing the square ⇔ Gaussian elimination Principle Axis Theorem Nutshell Optional material

Lecture 22/25: Positive Definite Matrices (PDMs)

Positive Definite

Motivation. What a PDM is





Outline

Positive Definite Matrices Motivation...

Lecture 22/25:

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Motivation...

What a PDM is... Identifying PDMs





What does this function look like?:

$$f(x_1,x_2)=2x_1^2-2x_1x_2+2x_2^2.$$

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Positive Definite Matrices (PDMs)

Positive Definite Matrices

Motivation...

What a PDM is...





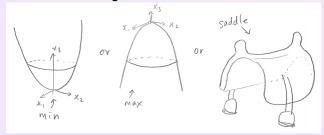




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► Three main categories:



- Standard approach for determining type of extremum involves calculus, derivatives, horrible things...
- ▶ Obviously, we should be using linear algebra...

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Identifying PDMe

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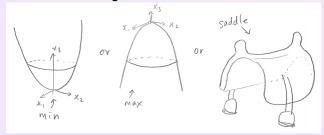




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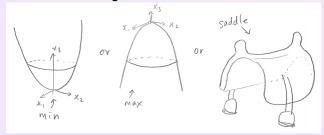




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Lecture 22/25: Positive Definite Matrices (PDMs)

Positive Definite Motivation...

What a PDM is

Linear Algebra-ization...

We can rewrite

$$f(x_1,x_2)=2x_1^2-2x_1x_2+2x_2^2.$$

as

$$f(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \boxed{\vec{x}^T \land \vec{x}}$$

- Note: A is symmetric as $A = A^{T}$ (delicious).
- Interesting and sneaky...







Lecture 22/25: Positive Definite Matrices (PDMs)

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Positive Definite Motivation...

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Positive Definite Matrices

Motivation...

What a PDM is...

Identifying PDMs

Completing the square ⇔
Gaussian elimination

Nutshall

Optional materi







What about this curve?:

$$2x_1^2 + 2x_1x_2 + 2x_2^2 = 1.$$

$$1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{\vec{x}^T \mathbb{A} \vec{x}}$$

Lecture 22/25: Positive Definite Matrices (PDMs)

Positive Definite Matrices

Motivation...

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Goal:

- ▶ Understand how \mathbb{A} governs the form $\vec{x}^T \mathbb{A} \vec{x}$.
- ► Somehow, this understanding will involve almost everything we've learnt so far: row reduction, pivots, eigenthings, symmetry, ...

Positive Definite Matrices

Motivation... What a PDM is

What a PDM is...

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Lecture 22/25: Positive Definite

Positive Definite

Motivation...
What a PDM is...

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Completing the square \Leftarrow Gaussian elimination

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Lecture 22/25: Positive Definite Matrices (PDMs)

Write $\mathbb{A} = \mathbb{A}^{\mathrm{T}} = \begin{vmatrix} a & b \\ b & c \end{vmatrix}$.

$$\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 \\ bx_2 + ax_2 \end{bmatrix}$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2$$

▶ See how a, b, and c end up in the quadratic form.



Motivation...

What a PDM is







Lecture 22/25:
Positive Definite
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Matrices

Motivation...
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Identifying PDMs
Completing the square ⇔

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$$= x_1(ax_1+bx_2)+x_2(bx_1+cx_2) = ax_1^2+bx_1x_2+bx_1x_2+c$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2.$$

► See how *a*, *b*, and *c* end up in the quadratic form.





Lecture 22/25: Positive Definite Matrices (PDMs)

Positive Definite

Motivation... What a PDM is Identifying PDMs

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Lecture 22/25:
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$$=ax_1^2+2bx_1x_2+cx_2^2.$$

▶ See how *a*, *b*, and *c* end up in the quadratic form.

Positive Definite

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Lecture 22/25: Positive Definite Matrices (PDMs)

Positive Definite

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Motivation... What a PDM is Identifying PDMs

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▶ See how a, b, and c end up in the quadratic form.







Lecture 22/25:
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Positive Definite

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Motivation...
What a PDM is...
Identifying PDMs
Completing the square ⇔
Gaussian elimination

$$\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}$$

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► See how a, b, and c end up in the guadratic form.







Lecture 22/25:
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Matrices (PDMs)

Positive Definite

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Motivation...
What a PDM is...
Identifying PDMs
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► See how *a*, *b*, and *c* end up in the quadratic form.







We have:
$$\vec{x}^T \mathbb{A} \vec{x} = ax_1^2 + 2bx_1x_2 + cx_2^2 = f(x_1, x_2)$$

Positive Definite Matrices (PDMs)

Lecture 22/25:

Positive Definite

Motivation...

What a PDM is







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Back to our first example: $f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2$.

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Identify a = 2, b = -1, and c = 2.

Lecture 22/25:
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Second example: $2x_1^2 + 2x_1x_2 + 2x_2^2 = 1$.

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Write
$$\mathbb{A} = \mathbb{A}^{T} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$
.

$$= ax_1^2 + dx_2^2 + fx_3^2 + 2bx_1x_2 + 2cx_1x_3 + 2ex_2x_3$$

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Positive Definite

Motivation...

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General 3×3 example:

Write
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$$= ax_1^2 + dx_2^2 + fx_3^2 + 2bx_1x_2 + 2cx_1x_3 + 2ex_2x_3.$$

 Again: see how the terms in A distribute into the euachalic form. Lecture 22/25:
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General story:

Using the definition of matrix multiplication,

$$\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x}$$

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General story:

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- ▶ We see the $x_i x_i$ term is attached to a_{ii} .



Motivation...

What a PDM is





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- ▶ We see the $x_i x_i$ term is attached to a_{ii} .
- ▶ On-diagonal terms look like this: $a_{77}x_7^2$ and $a_{33}x_3^2$.

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$$\vec{x}^{T} \mathbb{A} \vec{x} = \sum_{i=1}^{n} [\vec{x}^{T}]_{i} [\mathbb{A} \vec{x}]_{i} = \sum_{i=1}^{n} x_{i} \sum_{j=1}^{n} a_{ij} x_{j} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{i} x_{j}$$

- ▶ We see the $x_i x_i$ term is attached to a_{ii} .
- ▶ On-diagonal terms look like this: $a_{77}x_7^2$ and $a_{33}x_3^2$.
- ▶ Off-diagonal terms combine, e.g., $(a_{13} + a_{31})x_1x_3$.



Motivation...

What a PDM is





$$\vec{x}^{T} \mathbb{A} \vec{x} = \sum_{i=1}^{n} [\vec{x}^{T}]_{i} [\mathbb{A} \vec{x}]_{i} = \sum_{i=1}^{n} x_{i} \sum_{j=1}^{n} a_{ij} x_{j} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{i} x_{j}$$

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- ▶ Given some f with a term $23x_1x_3$, we could divide the 23 between a_{13} and a_{31} however we like.

Positive Definite Matrices

Motivation...

What a PDM is





Using the definition of matrix multiplication,

$$\vec{x}^{T} \mathbb{A} \vec{x} = \sum_{i=1}^{n} [\vec{x}^{T}]_{i} [\mathbb{A} \vec{x}]_{i} = \sum_{i=1}^{n} x_{i} \sum_{j=1}^{n} a_{ij} x_{j} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{i} x_{j}$$

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- e.g., $a_{13} = 36$ and $a_{31} = -13$ would work.

Positive Definite Matrices

Motivation... What a PDM is







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- $\vec{x}^{T} \mathbb{A} \vec{x} = \sum_{i=1}^{n} [\vec{x}^{T}]_{i} [\mathbb{A} \vec{x}]_{i} = \sum_{i=1}^{n} x_{i} \sum_{j=1}^{n} a_{ij} x_{j} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{i} x_{j}$
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- Given some f with a term $23x_1x_3$, we could divide the 23 between a_{13} and a_{31} however we like.
- e.g., $a_{13} = 36$ and $a_{31} = -13$ would work.
- ▶ But we choose to make A symmetric because symmetry is great.





A little abstraction:

Lecture 22/25:

Positive Definite Matrices (PDMs)

A few observations:

- 1. The construction $\vec{x}^T \mathbb{A} \vec{x}$ appears naturally.

$$\vec{v}^{\mathrm{T}} \mathbb{A} \vec{v} = \vec{v}^{\mathrm{T}} (\mathbb{A} \vec{v}) = \vec{v}^{\mathrm{T}} (\lambda \vec{v}) = \lambda \vec{v}^{\mathrm{T}} \vec{v} = \lambda ||\vec{v}||^2$$

Positive Definite

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- 1. The construction $\vec{x}^T \mathbb{A} \vec{x}$ appears naturally.
- 2. Dimensions of \vec{x}^T , \mathbb{A} , and \vec{x} : 1 by n, n by n, and n by 1.
- 3. $\vec{X}^T \mathbb{A} \vec{X}$ is a 1 by 1.
- 4. If $\mathbb{A}\vec{v} = \lambda \vec{v}$ then

$$\vec{v}^{\mathrm{T}} \mathbb{A} \vec{v} = \vec{v}^{\mathrm{T}} (\mathbb{A} \vec{v}) = \vec{v}^{\mathrm{T}} (\lambda \vec{v}) = \lambda \vec{v}^{\mathrm{T}} \vec{v} = \lambda ||\vec{v}||^2$$

- 5. If $\lambda > 0$, then $\vec{v}^{T} \mathbb{A} \vec{v} > 0$ always (given $\vec{v} \neq \vec{0}$).
- 6. Suggests we can build up to saying something about $\vec{x}^T \mathbb{A} \vec{x}$ starting from eigenvalues...

Positive Definite Matrices

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Positive Definite

Motivation...

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Positive Definite

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Definitions:

Positive Definite Matrices (PDMs):

- ► Real, symmetric matrices with positive eigenvalues.
- Math version:

$$\mathbb{A}=\mathbb{A}^{\mathrm{T}},$$
 $a_{ij}\in R\ orall\ i,j=1,2,\cdots n,$ and $\lambda_i>0,\ orall\ i=1,2,\cdots n$

Semi-Positive Definite Matrices (SPDMs):

Same as for PDMs but now eigenvalues may now be 0:

$$\lambda_i \geq 0, \ \forall \ i=1,2,\cdots,n.$$

Note: If some eigenvalues are < 0 we have a sneaky matrix Lecture 22/25:

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Equivalent Definitions:

Lecture 22/25: Positive Definite Matrices (PDMs)

Positive Definite Matrices:

 $\blacktriangle = \mathbb{A}^{T}$ is a PDM if

$$\vec{x}^{\mathrm{T}} \mathbb{A} \, \vec{x} > 0 \ \forall \ \vec{x} \neq \vec{0}$$

Positive Definite Motivation... What a PDM is...







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Connecting these definitions:

Lecture 22/25: Positive Definite Matrices (PDMs)

Spectral Theorem for Symmetric Matrices:

$$\boxed{\mathbb{A} = \mathbb{Q} \, \Lambda \, \mathbb{Q}^T}$$

where $\mathbb{Q}^{-1} = \mathbb{Q}^T$,

$$\mathbb{Q} = \begin{bmatrix} | & | & \cdots & | \\ \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \\ | & | & \cdots & | \end{bmatrix}, \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

▶ Special form of $\mathbb{A} = \mathbb{S}\Lambda\mathbb{S}^{-1}$ that arises when $\mathbb{A} = \mathbb{A}^T$.

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▶ Substitute $\mathbb{A} = \mathbb{Q} \wedge \mathbb{Q}^{T}$ into $\vec{x}^{T} \mathbb{A} \vec{x}$:

 $= (\mathbb{Q}^{\mathrm{T}}\vec{x})^{\mathrm{T}} \wedge (\mathbb{Q}^{\mathrm{T}}\vec{x})$

We now see \vec{x} transforming from the natural basis to \mathbb{A} 's eigenvector basis: $\vec{v} = \mathbb{Q}^T \vec{x}$.

$$: \vec{X}^{\mathrm{T}} \mathbb{A} \, \vec{X} = \vec{Y}^{\mathrm{T}} \wedge \vec{Y}$$

$$= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2.$$

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$$=\lambda_1y_1^2+\lambda_2y_2^2+\cdots+\lambda_ny_n^2.$$

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Matrices (PDMs)

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So now we have...

$$\vec{x}^{\mathrm{T}} \mathbb{A} \, \vec{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

- ► Can see whether or not $\vec{x}^T \mathbb{A} \vec{x} > 0$ depends on the λ_i since each $y_i^2 > 0$.
- ▶ So a PDM must have each $\lambda_i > 0$.
- ▶ And a SPDM must have $\lambda_i \geq 0$.

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What a PDM is...

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Gaussian elimination
Principle Avis Theorem

Nutsneii





More understanding of $\vec{x}^T \mathbb{A} \vec{x}$:

▶ Substitute $\mathbb{A} = \mathbb{L} \mathbb{D} \mathbb{L}^T$ into $\vec{x}^T \mathbb{A} \vec{x}$:

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Lecture 22/25:

Positive Definite
Matrices (PDMs)

Positive Definite Matrices

Matrices

Motivation...
What a PDM is...

What a PDM is...
Identifying PDMs

Completing the square
Gaussian elimination
Principle Axis Theorem

Nutshell

Ориона шавна







More understanding of $\vec{x}^T \mathbb{A} \vec{x}$:

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Lecture 22/25: Positive Definite Matrices (PDMs)

Positive Definite

What a PDM is...







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Lecture 22/25: Positive Definite

Matrices (PDMs)

Positive Definite

What a PDM is... Identifying PDMs







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Lecture 22/25:
Positive Definite
Matrices (PDMs)

Positive Definite

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What a PDM is...

Identifying PDMs

Completing the square ⇔

Gaussian elimination

Nutshell







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Lecture 22/25: Positive Definite Matrices (PDMs)

Positive Definite

What a PDM is... Identifying PDMs









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Positive Definite

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Lecture 22/25:
Positive Definite
Matrices (PDMs)

Positive Definite Matrices

Matrices

Motivation...

What a PDM is...

Completing the square
Gaussian elimination

Nutshell







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Lecture 22/25:
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Lecture 22/25:

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Positive Definite Matrices (PDMs)

So now we have...

$$\vec{x}^{\mathrm{T}} \mathbb{A} \, \vec{x} = d_1 z_1^2 + d_2 z_2^2 + \dots + d_n z_n^2$$

- ► Can see whether or not $\vec{x}^T \mathbb{A} \vec{x} > 0$ depends on the d_i
- ▶ So a PDM must have each $d_i > 0$.
- ▶ And a SPDM must have $d_i > 0$.

What a PDM is...







Lecture 22/25:

Positive Definite

Positive Definite Matrices (PDMs)

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Matrices
Motivation...
What a PDM is...
Identifying PDMs

Identifying PDMs

Completing the square
Gaussian elimination

Principle Axis Theorem





Lecture 22/25:

Positive Definite Matrices (PDMs)

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Lecture 22/25:

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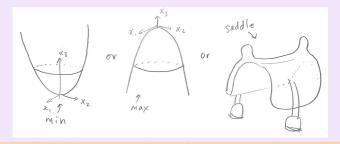
What a PDM is...

Positive Definite





$$f(x,y) = \vec{x}^{\mathrm{T}} \mathbb{A} \vec{x} = ax_1^2 + 2bx_1x_2 + cx_2^2$$



Focus on eigenvalues—We can now see:

- ▶ f(x, y) has a minimum at x = y = 0 iff A is a PDM, i.e., if $\lambda_1 > 0$ and $\lambda_2 > 0$.
- ▶ Maximum: if $\lambda_1 < 0$ and $\lambda_2 < 0$.
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Lecture 22/25:

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Positive Definite

Matrices

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Identifying PDMs

Completing the square

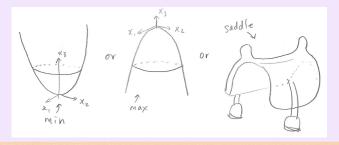
Principle Axis Theorem Nutshell







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Lecture 22/25:

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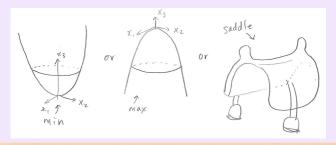
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Lecture 22/25:

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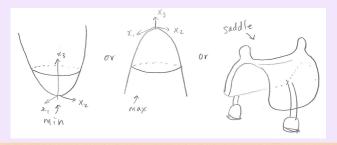
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Lecture 22/25:

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Motivation...

What a PDM is...

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Gaussian elimination

Principle Axis Theorem

Nutshell







Back to simple example problem 1 of 2:

$f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Matrices

Motivation...

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Optional materia

Compute eigenvalues...

Find $\lambda_1 = +3$ and $\lambda_2 = +1$: f is a minimum.

General problem

▶ How do we easily find the signs of λ s...?







Back to simple example problem 1 of 2:

Positive Definite

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What a PDM is... Identifying PDMs

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Matrices

Motivation...

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Ontional mater

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Outline

Positive Definite Matrices

Identifying PDMs

Lecture 22/25: Positive Definite Matrices (PDMs)

Positive Definite

Motivation... What a PDM is





- We recall with alacrity the totally amazing fact that real symmetric matrices always have (1) real eigenvalues, and (2) orthogonal eigenvectors forming a basis for R^n .
- ▶ We now see that knowing the signs of the λ s is also

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Test cases:

Lecture 22/25:

Positive Definite Matrices (PDMs)

Positive Definite

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Test cases:

Some minor struggling leads to:

- \blacktriangle $\Lambda_1: \lambda_1 = +3, \lambda_2 = +1, (PDM, happy),$
- $\mathbb{A}_2: \lambda_1 = +\sqrt{5}, \lambda_2 = -\sqrt{5}, \text{ (sad)},$

Lecture 22/25:

Positive Definite Matrices (PDMs)

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What a PDM is







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Lecture 22/25:

Positive Definite Matrices (PDMs)

Positive Definite Matrices

Motivation... What a PDM is..

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Lecture 22/25:

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Lecture 22/25:

Positive Definite Matrices (PDMs)

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Motivation... What a PDM is..

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Lecture 22/25: Positive Definite Matrices (PDMs)

Positive Definite Matrices

What a PDM is





Lecture 22/25: Positive Definite Matrices (PDMs)

Extremely Sneaky Result #632:

If $\mathbb{A} = \mathbb{A}^T$ and \mathbb{A} is real, then

- # +ve eigenvalues = # +ve pivots
- ▶ # -ve eigenvalues = # -ve pivots
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Positive Definite

What a PDM is...





Lecture 22/25: Positive Definite Matrices (PDMs)

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Positive Definite

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Lecture 22/25:
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- Previously, we had for general A that $|\mathbb{A}| = \prod \lambda_i = \pm \prod d_i$.
- ► The bonus here is for real symmetric A.
- Eigenvalues are pivots come from very different parts of linear algebra.
- Crazy connection between eigenvalues and pivots!

Positive Definite

What a PDM is







Pivots and Eigenvalues:

Lecture 22/25: Positive Definite Matrices (PDMs)

More notes:

- All very exciting: Pivots are much, much easier to
- (cue balloons, streamers)

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More notes:

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Check for our three examples:

- \blacktriangle $A_1: d_1 = +2, d_2 = +\frac{3}{2}$ \checkmark signs match with $\lambda_1 = +3, \lambda_2 = +1$.
- $A_2: d_1 = +2, d_2 = -\frac{5}{2}$
- $A_3: d_1 = -2, d_2 = -\frac{3}{2}$

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- \blacktriangle $\mathbb{A}_2: d_1 = +2, d_2 = -\frac{5}{2}$ \checkmark signs match with $\lambda_1 = +\sqrt{5}, \lambda_2 = -\sqrt{5}$.
- $\mathbb{A}_3: d_1=-2, d_2=-\frac{3}{2}$

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- $ightharpoonup A_3: d_1 = -2, d_2 = -\frac{3}{2}$ \checkmark signs match with $\lambda_1 = -1, \lambda_2 = -3$.

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Let's show how the signs of eigenvalues match signs of pivots for

$$\mathbb{A}_2 = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}, \quad \lambda_{1,2} = \pm \sqrt{5}$$

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▶ Compute LU decomposition:

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & -\frac{5}{2} \end{bmatrix} = \mathbb{L}\mathbb{U}$$

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Motivation... What a PDM is...

Identifying PDMs

Completing the square ⇔ Gaussian elimination

Nutshell





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A₂ is symmetric, so we can go further:

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} = \mathbb{LDL}^{\mathbb{T}}$$

Positive Definite

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Now think about this matrix:

$$\mathbb{B}(\ell_{21}) = \begin{bmatrix} 1 & 0 \\ \ell_{21} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & \ell_{21} \\ 0 & 1 \end{bmatrix}$$

▶ When $\ell_{21} = -\frac{1}{2}$, we have $B(-\frac{1}{2}) = \mathbb{A}_2$.

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Positive Definite Matrices

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What a PDM is...

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- ▶ When $\ell_{21} = -\frac{1}{2}$, we have $B(-\frac{1}{2}) = \mathbb{A}_2$.
- ▶ Think about what happens as ℓ_{21} changes smoothly from $-\frac{1}{2}$ to 0.

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▶ We're here:

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Positive Definite Matrices

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- 1. $\mathbb{B}(0) = \mathbb{D}$'s eigenvalues and pivots are both 2, $-\frac{5}{2}$.
- 2. Stronger: As we alter $\mathbb{B}(\mathbb{F}_{21})$, the pivots do not change!
- 3. But eigenvalues do change from $\pm\sqrt{5}$ and $-\sqrt{5}$ to
- 4. Big deal: because the pivots don't change, the determinant of $\mathbb{B}(\ell_{2})$ never changes:
 - $\det \mathbb{B}(\ell_{21}) = d_1 \cdot d_2 = 2 \cdot \left(-\frac{5}{2}\right) = -5 \neq 0$
- 5. But we also know $\det \mathbb{B}(\ell_{21}) = \lambda_1 + \lambda_2$
- 6. as ℓ_2 ; changes, eigenvalues cannot pass through
- 7. eigenvalues cannot change sign as ℓ_{21} changes.
- 8. Signs of eigenvalues of $\mathbb{A}_2 = \mathbb{B}(-\frac{1}{2})$ must match signs of $\mathbb{B}(0)$ which match signs of
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Can see argument extends to n by n's.

• Take $\mathbb{A} = \mathbb{A}^1 = \mathbb{L} \mathbb{D} \mathbb{L}^1$ and smoothly change \mathbb{L} to

ightharpoonup Write $\mathbb{L}(t) = \mathbb{I} + t(\mathbb{L} - \mathbb{I})$ and

 $\mathbb{B}(t) = \hat{\mathbb{L}}(t) \, \mathbb{D} \, \hat{\mathbb{L}}(t)^{\mathrm{T}}$

- ➤ When t = 1, we have $\mathbb{L}(1) = \mathbb{L}$ and $\mathbb{B}(1) = \mathbb{L}$
- \blacktriangleright When t=0. L(0) = I. and B(0)
- ► Again, pivots don't change as we move t from 1 to 0, and determinant must stay the same.
- Same story, eigenvalues cannot cross zero and mus have the same signs for all t, including t = 0 when eigenvalues and pivots are equal x = 0.

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Principle Axis Theorem Nutshell





- Can see argument extends to n by n's.
- ▶ Take $A = A^T = \mathbb{LDL}^T$ and smoothly change $\mathbb L$ to $\mathbb I$.

ightharpoonup Write $\mathbb{L}(t) = \mathbb{I} + t(\mathbb{L} - \mathbb{I})$ and

• When t=1, we have $\mathbb{L}(1)=\mathbb{L}$ and

 $P = \{ (0, \mathbb{R}(0)) = 1, \text{ and } \mathbb{R}(0) \}$

and determinant must stay the same

have the same signs for all t, including t=0 when

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Optional material





- Can see argument extends to n by n's.
- ▶ Take $\mathbb{A} = \mathbb{A}^{T} = \mathbb{LDL}^{T}$ and smoothly change \mathbb{L} to \mathbb{I} .
- Write $\hat{\mathbb{L}}(t) = \mathbb{I} + t(\mathbb{L} \mathbb{I})$ and

$$\mathbb{B}(t) = \hat{\mathbb{L}}(t) \, \mathbb{D} \, \hat{\mathbb{L}}(t)^{\mathrm{T}}$$

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- When t = 0, $\hat{\mathbb{L}}(0) = \mathbb{I}$, and $\mathbb{B}(0) = \mathbb{D}$.

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Outline

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'Complete the square' for our first example:

$$f(x_1,x_2)=2x_1^2-2x_1x_2+2x_2^2$$

$$=2(x_1^2-x_1x_2)+2x_2^2=2(x_1^2-x_1x_2+\frac{1}{4}x_2^2-\frac{1}{4}x_2^2)+2x_2^2$$

$$=2(x_1^2-x_1x_2+\frac{1}{4}x_2^2)-\frac{1}{2}x_2^2+2x_2^2=2(x_1-\frac{1}{2}x_2)^2+\frac{3}{2}x_2^2$$

We see the pivots $d_1 = 2$ and $d_2 = \frac{3}{2}$ and the multiplier $\ell_{21} = -\frac{1}{2}$ appear:

$$f(x_1,x_2)=d_1(x_1+\ell_{21}x_2)^2+d_2x_2^2.$$

- Super cool—this is exactly $\vec{x}^T \mathbb{A} \vec{x} = (\mathbb{L}^T \vec{x}) \mathbb{D} (\mathbb{L}^T \vec{x})^T = d_1 z_1^2 + d_2 z_2^2$.
- ▶ The minimum is now obvious (sum of squares).

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$$=2(x_1^2-x_1x_2+\frac{1}{4}x_2^2)-\frac{1}{2}x_2^2+2x_2^2=2(x_1-\frac{1}{2}x_2)^2+\frac{3}{2}x_2^2$$

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Lecture 22/25:

Positive Definite Matrices (PDMs)

Positive Definite Matrices

Motivation...
What a PDM is...
Identifying PDMs

Completing the square ⇔ Gaussian elimination

Principle Axis Theorem
Nutshell







'Complete the square' for our first example:

$$f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2$$

$$= 2(x_1^2 - x_1x_2) + 2x_2^2 = 2(x_1^2 - x_1x_2 + \frac{1}{4}x_2^2 - \frac{1}{4}x_2^2) + 2x_2^2$$

$$=2(x_1^2-x_1x_2+\frac{1}{4}x_2^2)-\frac{1}{2}x_2^2+2x_2^2=2(x_1-\frac{1}{2}x_2)^2+\frac{3}{2}x_2^2$$

We see the pivots $d_1 = 2$ and $d_2 = \frac{3}{2}$ and the multiplier $\ell_{21} = -\frac{1}{2}$ appear:

$$f(x_1,x_2)=d_1(x_1+\ell_{21}x_2)^2+d_2x_2^2.$$

- ▶ Super cool—this is exactly $\vec{x}^T \mathbb{A} \vec{x} = (\mathbb{L}^T \vec{x}) \mathbb{D} (\mathbb{L}^T \vec{x})^T = d_1 z_1^2 + d_2 z_2^2$.
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Another example:

► Take the matrix A₂:

$$f(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Complete the square:

Matches Pyots $d_1 = 2$, $d_2 = -\frac{5}{2}$, so $x_1 = x_2 = 0$ is a

Completing the square matches up with elimination

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Back to our second simple problem:

- Graph $2x_1^2 + 2x_1x_2 + 2x_2^2 = 1$.
- We'll simplify with linear algebra to find an equation of an ellipse...
- From before, our equation can be rewritten as

$$\vec{X}^{\mathrm{T}} \mathbb{A} \, \vec{X} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

Again use spectral decomposition, $\mathbb{A} = \mathbb{Q} \wedge \mathbb{Q}^{T}$, to diagonalize giving $(\mathbb{Q}^{T}\vec{x})^{T} \wedge (\mathbb{Q}^{T}\vec{x}) = 1$ where

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So
$$2x_1^2 + 2x_1x_2 + 2x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

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crazily becomes

$$\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right)^{\mathrm{T}} = 1$$

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$$: \begin{bmatrix} \frac{x_1 + x_2}{\sqrt{2}} & \frac{x_1 - x_2}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x_1 + x_2}{\sqrt{2}} \\ \frac{x_1 - x_2}{\sqrt{2}} \end{bmatrix} = 1$$

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$$3\left(\frac{x_1+x_2}{\sqrt{2}}\right)^2+\left(\frac{x_1-x_2}{\sqrt{2}}\right)^2=1$$

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If we change to eigenvector coordinate system,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbb{Q}^{\mathrm{T}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1 + x_2}{\sqrt{2}} \\ \frac{x_1 - x_2}{\sqrt{2}} \end{bmatrix},$$

then our equation simplifies greatly:

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 1,$$

which is just

$$3 \cdot u_1^2 + 1 \cdot u_2^2 = 1.$$

Very nice! PDM : ellipse.

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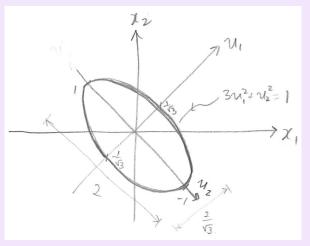
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Finally, we can draw a picture of $2x_1^2 + 2x_1x_2 + 2x_2^2$:



 $3 \cdot u_1^2 + 1 \cdot u_2^2 = 1$ where $u_1 = \frac{x_1 + x_2}{\sqrt{2}}$ and $u_2 = \frac{x_1 - x_2}{\sqrt{2}}$.

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- $\vec{x}^{T} \mathbb{A} \vec{x}$ is a commonly occurring construction.
- Big deals: Positive Definiteness and Semi-Positive Definiteness of A
- ▶ Positive eigenvalues : PDM
- ▶ Non-negative eigenvalues : SPDM
- Signs of pivots (easy test) match signs of eigenvalues.
- ► Gaussian elimination = completing the square
- Standard questions: determine if a matrix is a PDM, convert a quadratic function into matrix \$\overline{x}^1 \times \overline{x}\$, sketch a quadratic curve (e.g., an ellipse).

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Standard questions: determine if a matrix is a PDM,





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ST #731:



For a real symmetric A, if all upper left determinants of A are +ve, so are A's eigenvalues, and vice versa.







Lecture 22/25: Positive Definite Matrices (PDMs)

ST #731:



For a real symmetric A, if all upper left determinants of A are +ve, so are A's eigenvalues, and vice versa.

Check:

$$All A_1: |2| > 0, \quad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0: yes.$$

$$Arr A_2: |2| > 0, \quad \begin{vmatrix} 2 & -1 \\ -1 & -2 \end{vmatrix} = -5 < 0: \text{no}.$$

$$ightharpoonup \mathbb{A}_3: |-2| < 0, \ \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = 3 > 0: \text{no}.$$

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ST #731:



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- ► Take general symmetric matrix 2 × 2: $\mathbb{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$
- ► Upper left determinants: a and ac b
- Eigenvalues (from Assignment 9)
 - $\lambda_1 = \frac{(a+c) + \sqrt{(a-c)^2 + 4b}}{2}$
 - $\frac{(a+c)-\sqrt{(a-c)^2+4b^2}}{2}$
- Objective
 - show a>0 and $ac-b^2>0\Rightarrow \lambda_1,\lambda_2>0$

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Reuse previous sneakiness:

$$|A - \lambda I| = \begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = (a - \lambda)(c - \lambda) - b^{2}$$

$$= \lambda^{2} - (a + c)\lambda + ac - b^{2}$$

$$= \lambda^{2} - \text{Tr}(A) + \text{det}(A)$$

$$= \lambda^{2} - (\lambda_{1} + \lambda_{2}) + (\lambda_{1} \cdot \lambda_{2})$$

$$:\lambda_1 + \lambda_2 = a + c, \quad \lambda_1 \cdot \lambda_2 = ac - b^2$$

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Reasoning for 2×2 case:

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Show ":":

- ▶ Given $ac b^2 > 0$ then $\lambda_1 \cdot \lambda_2 > 0$, so both eigenvalues are positive or both are negative
- ▶ Given a > 0 then c > 0 b/c otherwise $ac b^2 < 0$.
- ▶ This means $a + c = \lambda_1 + \lambda_2 > 0$ → both eigenvalues are positive.

Show "←":

- ▶ Given λ_1 , $\lambda_2 > 0$, then $ac b^2 = \lambda_1 \cdot \lambda_2 > 0$
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Upshot: We can compute determinants instead of eigenvalues to find signs.

But: Computing determinants still isn't a picnic either...

- ► A much better way is to use the connection between pivots and eigenvalues.
- Another weird connection.

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