Chapter 3/4: Lecture 15

Matrixology (Linear Algebra)—Lecture 14/25
MATH 124, Fall, 2011

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont

Outline

Review for Exam 2

Words

Pictures

Basics:

Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1–4.3)
- Main pieces:
  1. Big Picture of $\mathbb{A}\vec{x} = \vec{b}$
  2. Projections and the normal equation
- As always, want 'doing' and 'understanding' abilities.

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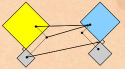

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- Chapter 3 and Chapter 4 (Sections 4.1–4.3)
- Main pieces:
  1. Big Picture of $\mathbb{A}\vec{x} = \vec{b}$
     Must be able to draw the big picture!
  2. Projections and the normal equation
- As always, want 'doing' and 'understanding' abilities.

Stuff to know/understand

Vector Spaces:

- Vector space concept and definition.
- Subspace definition (three conditions).
- Concept of a spanning set of vectors.
- Concept of a basis.
- Basis = minimal spanning set.
- Concept of orthogonal complement.
- Various techniques for finding bases and orthogonal complements.

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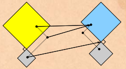

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Fundamental Theorem of Linear Algebra:

- Applies to any $m \times n$ matrix $\mathbb{A}$.
- Symmetry of $\mathbb{A}$ and $\mathbb{A}^{\mathrm{T}}$.

- Column space $C(\mathbb{A}) \subset R^m$.
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- $\dim C(\mathbb{A}) + \dim N(\mathbb{A}^{\mathrm{T}}) = r + (m - r) = m$
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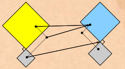

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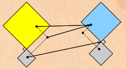

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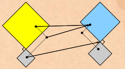

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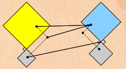

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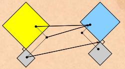

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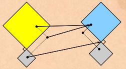

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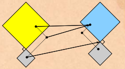

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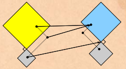
Finding four fundamental subspaces:

- Enough to find bases for subspaces.
- Be able to reduce $\mathbb{A}$ to $\mathbb{R}_{\mathbb{A}}$ and $\mathbb{A}^{\mathrm{T}}$ to $\mathbb{R}_{\mathbb{A}^{\mathrm{T}}}$.
- Understand crucial nature of $\mathbb{R}_{\mathbb{A}}$ and $\mathbb{R}_{\mathbb{A}^{\mathrm{T}}}$.
- Identify pivot columns and free columns.
- Rank r of $\mathbb{A}$ = # pivot columns.
- Know that relationship between $\mathbb{R}_{\mathbb{A}}$'s columns hold for $\mathbb{A}$'s columns.



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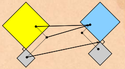

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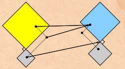

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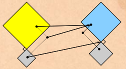

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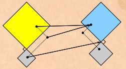

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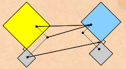

Stuff to know/understand:

Bases for column space—three ways:

1. Find when $\mathbb{A}\vec{x} = \vec{b}$ has a solution:
   - Reduce $[\mathbb{A} \,|\, \vec{b}]$ where $\vec{b}$ is general.
   - Find conditions on $\vec{b}$'s elements for a solution to $\mathbb{A}\vec{x} = \vec{b}$ to exist $\rightarrow$ obtain basis for $C(\mathbb{A})$.
2. Use $\mathbb{R}_{\mathbb{A}}$:
   - Find pivot columns in $\mathbb{R}_{\mathbb{A}}$—same columns in $\mathbb{A}$ form a basis for $C(\mathbb{A})$.
   - Warning: $\mathbb{R}_{\mathbb{A}}$'s columns do not give a basis for $C(\mathbb{A})$
3. Use $\mathbb{R}_{\mathbb{A}^{\mathrm{T}}}$:
   - Best and easiest way: basis for column space = non-zero rows in $\mathbb{R}_{\mathbb{A}^{\mathrm{T}}}$, the reduced form of $\mathbb{A}^{\mathrm{T}}$.

Basis for row space:

- Take non-zero rows in $\mathbb{R}_{\mathbb{A}}$ (easy!).
- Matches way 3 for column space.

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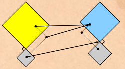

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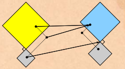

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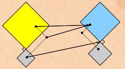

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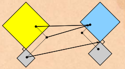

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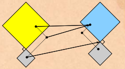

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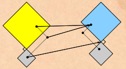

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- Basis for nullspace obtained by solving $\mathbb{A}\vec{x} = \vec{0}$
- Always express pivot variables in terms of free variables.
- Free variables are unconstrained (can be any real number)
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- Similarly find basis for $N(\mathbb{A}^{\mathrm{T}})$ by solving $\mathbb{A}^{\mathrm{T}}\vec{y} = \vec{0}$.
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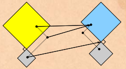

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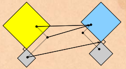

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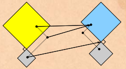

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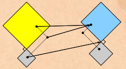

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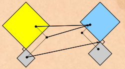

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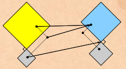

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Number of solutions to $\mathbb{A}\vec{x} = \vec{b}$:

1. If $\vec{b} \notin C(\mathbb{A})$, there are no solutions.
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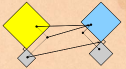

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Projections:

- Understand how to project a vector $\vec{b}$ onto a line in direction of $\vec{a}$.
- $\vec{b} = \vec{p} + \vec{e}$
- $\vec{p}$ = that part of $\vec{b}$ that lies in the line:

$$\vec{p} = \frac{\vec{a}^{\mathrm{T}}\vec{b}}{\vec{a}^{\mathrm{T}}\vec{a}}\vec{a} \left(= \frac{\vec{a}\vec{a}^{\mathrm{T}}}{\vec{a}^{\mathrm{T}}\vec{a}}\vec{b} \right)$$

- $\vec{e}$ = that part of $\vec{b}$ that is orthogonal to the line.
- Understand generalization to projection onto subspaces.
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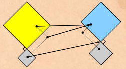

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Normal equation for $\mathbb{A}\vec{x} = \vec{b}$:

- If $\vec{b} \notin C(\mathbb{A})$, project $\vec{b}$ onto $C(\mathbb{A})$.
- Write projection of $\vec{b}$ as $\vec{p}$.
- Know $\vec{p} \in C(\mathbb{A})$ so $\exists\ \vec{x}_*$ such that $\mathbb{A}\vec{x}_* = \vec{p}$.
- Error vector must be orthogonal to column space so $\mathbb{A}^{\mathrm{T}}\vec{e} = \mathbb{A}^{\mathrm{T}}(\vec{b} - \vec{p}) = \vec{0}$.
- Rearrange:

$$\mathbb{A}^{\mathrm{T}}\vec{p} = \mathbb{A}^{\mathrm{T}}\vec{b}$$

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- Since $\mathbb{A}\vec{x}_* = \vec{p}$, we end up with

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- Error vector must be orthogonal to column space so $\mathbb{A}^{\mathrm{T}}\vec{e} = \mathbb{A}^{\mathrm{T}}(\vec{b} - \vec{p}) = \vec{0}$.
- Rearrange:

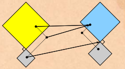
$$\mathbb{A}^{\mathrm{T}}\vec{p} = \mathbb{A}^{\mathrm{T}}\vec{b}$$

.

- Since $\mathbb{A}\vec{x}_* = \vec{p}$, we end up with

$$\mathbb{A}^{\mathrm{T}}\mathbb{A}\vec{x}_* = \mathbb{A}^{\mathrm{T}}\vec{b}.$$

- This is linear algebra's normal equation; $\vec{x}_*$ is our best solution to $\mathbb{A}\vec{x} = \vec{b}$.



Stuff to know/understand

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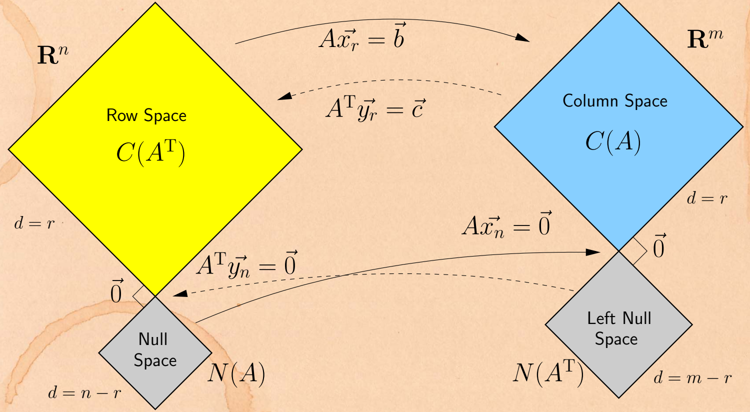
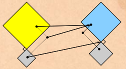
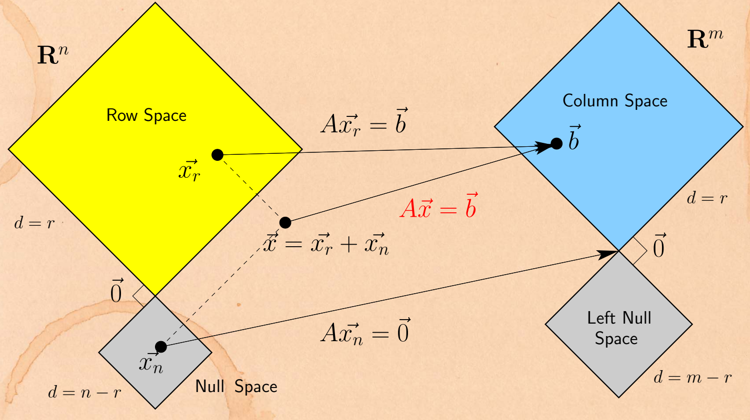
The symmetry of $\mathbb{A}\vec{x} = \vec{b}$ and $\mathbb{A}^{\mathrm{T}}\vec{y} = \vec{c}$:

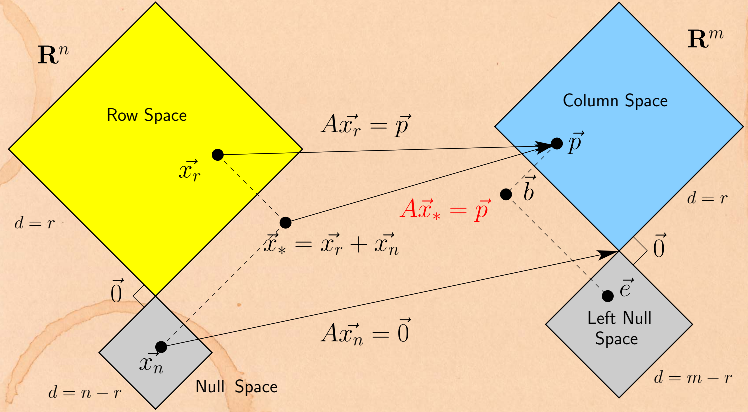


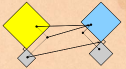


How $\mathbb{A}\vec{x} = \vec{b}$ works:

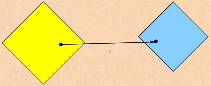
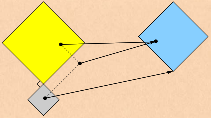
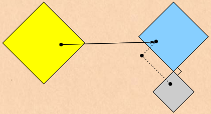
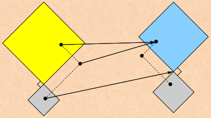


Best solution $\vec{x}_*$ when $\vec{b} = \vec{p} + \vec{e}$:





The fourfold ways of $\mathbb{A}\vec{x} = \vec{b}$:

case	example R	big picture	# solutions
$m = r$ $n = r$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		1 always
$m = r$, $n > r$	$\begin{bmatrix} 1 & 0 & \text{☕}_1 \\ 0 & 1 & \text{☕}_2 \end{bmatrix}$		∞ always
$m > r$, $n = r$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$		0 or 1
$m > r$, $n > r$	$\begin{bmatrix} 1 & 0 & \text{🚲}_1 \\ 0 & 1 & \text{🚲}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		0 or ∞