

Solving Linear Equations

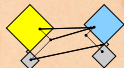
Matrixology (Linear Algebra)—Lecture 2/25

MATH 124, Fall, 2011

$$\text{Solving } A \vec{x} = \vec{b}$$

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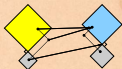


Outline

Lecture 2/25:
Solving Linear
Equations

Solving $A \vec{x} = \vec{b}$

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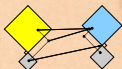


Solving $A\vec{x} = \vec{b}$:

- ▶ We (people + computers) solve systems of linear equations by a systematic method of **Elimination** followed by **Back substitution**
- ▶ Due to our man Gauss, hence Gaussian elimination.
- ▶ Our first example:

$$\begin{array}{rclcrcl} -x_1 & + & 3x_2 & = & 1 & & \\ 2x_1 & + & x_2 & = & 5 & & \end{array} \quad (1)$$

↪ chalkage



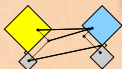
Gaussian elimination:

Basic elimination rules (roughly):

1. Strategically, mechanically remove unwanted entries by subtracting a multiple of a row from another.
2. Swap rows if needed to create an 'upper triangular form'

e.g.

$$\begin{array}{rcl} & x_2 = 3 & \\ 2x_1 - x_2 = -1 & \rightarrow & \begin{array}{r} 2x_1 - x_2 = -1 \\ x_2 = 3 \end{array} \end{array}$$



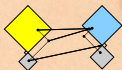
Gaussian elimination:

Solve:

$$2x_1 - 3x_2 = 3$$

$$4x_1 - 5x_2 + x_3 = 7$$

$$2x_1 - x_2 - 3x_3 = 5$$



Gaussian elimination:

Summary:

Using **row operations**, we turned this problem:

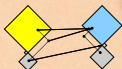
$$A\vec{x} = \vec{b} : \begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

into this problem:

$$U\vec{x} = \vec{d} : \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

and the latter is **easy to solve** using **back substitution**.

Solving $A\vec{x} = \vec{b}$



Defn:

The entries along U 's main diagonal are the **pivots** of A .
(The pivots are hidden—elimination finds them.)

Solving $A\vec{x} = \vec{b}$

Defn:

A matrix with only zeros below the main diagonal is called **upper triangular**. A matrix with only zeros above the main diagonal is called **lower triangular**. We travel from A to U and the latter is always upper triangular.

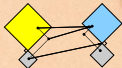
Defn:

Singular means a system has no unique solution.

- ▶ It may have no solutions or infinitely many solutions.
- ▶ Singular = archaic way of saying 'messed up.'

Truth:

If at least one pivot is zero, the matrix will be **singular**.
(but the reverse is not necessarily true).

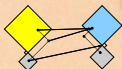


Gaussian elimination:

The one true method:

- ▶ We simplify A using elimination in **the same way every time**.
- ▶ Eliminate entries one column at a time, moving left to right, and down each column.

$$\begin{array}{ccccccccc} X & + & X & + & X & + & X & = & X \\ 1 \downarrow & + & X & + & X & + & X & = & X \\ 2 \downarrow & + & 4 \downarrow & + & X & + & X & = & X \\ 3 \nearrow & + & 5 \rightarrow & + & 6 & + & X & = & X \end{array}$$



Gaussian elimination:

Solving $A\vec{x} = \vec{b}$

- ▶ To eliminate entry in row i of j th column, subtract a multiple l_{ij} of the j th row from i .
- ▶ For example:

$$\begin{array}{rccccrcr} 2x_1 & + & 3x_2 & + & -2x_3 & + & x_4 & = & 1 \\ x_1 & - & 7x_2 & + & 3x_3 & + & x_4 & = & 1 \\ -x_1 & - & 3x_2 & - & x_3 & + & 5x_4 & = & -2 \\ 2x_1 & + & x_2 & - & 2x_3 & + & 2x_4 & = & 0 \end{array}$$

$$l_{21} = 1/2, l_{31} = -1/2, l_{41} = ?.$$

- ▶ **Note:** we cannot find l_{32} etc., until we are finished with row 1. Pivots are hidden!
- ▶ **Note:** the denominator of each l_{ij} multiplier is the pivot in the j th column.

