Introduction

Matrixology (Linear Algebra)—Lecture 1/25
MATH 124, Fall, 2011

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont

Outline

Exciting Admin

Importance

Usages

Key problems

Three ways of looking...

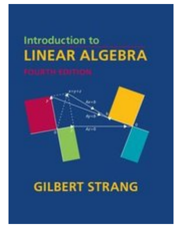
Colbert on Equations

References

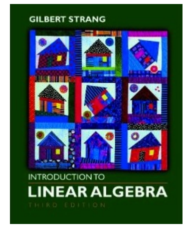
Basics:

- Instructor: Prof. Peter Dodds
- Lecture room and meeting times:
  254 Votey Hall,
  Tuesday and Thursday, 2:30 pm to 3:45 pm
- Office: Farrell Hall, second floor, Trinity Campus
- E-mail: peter.dodds@uvm.edu
- Course website: `http://www.uvm.edu/~pdodds/teaching/courses/2011-08UVM-124` (⊞)
- Textbook: "Introduction to Linear Algebra" (3rd of 4th editions) by Gilbert Strang (published by Wellesley-Cambridge Press).

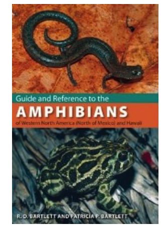
Our Textbook of Excellence:



4th Edition ☑



3rd Edition ☑



Unhelpful ☐

- "Introduction to Linear Algebra" by Gil Strang (⊞);
- Textbook website: http://math.mit.edu/linearalgebra/ (⊞)
- MIT Open Courseware site for 18.06 (=Linear Algebra): http://ocw.mit.edu/...linear-algebra-spring-2010/ (⊞)

Yesness:

Money quote from George Cobb's review of Strang's book:

Do you want a book written by a mathematician with a lifetime experience using linear algebra to understand important, authentic, applied problems, a former president of the Society for Industrial and Applied Mathematics,
or do you want a book shaped mainly by the [a]esthetics of pure mathematicians with only a weak, theoretical connection to how linear algebra is used in the natural and social sciences?

- George Cobb: Robert L. Rooke Professor of Mathematics and Statistics, Mount Holyoke College
- Full review here (⊞) [amazon]

Gil Strang, Exalted Friend of the Matrix:

- Professor of Mathematics at MIT since 1962.

These are 121 cupcakes with my favorite -1, 2, -1 matrix. It was the day before Thanksgiving and two days before my birthday. A happy surprise.

- Many awards including MAA Haimo Award (⊞) for Distinguished College or University Teaching of Mathematics
- Rhodes Scholar.
- Legend.

- More on Laplacian matrices, graphs, and other madnesses here (⊞).
- (Strang's Wikipedia page is here (⊞).

Admin:

Potential paper products:

1. Outline

Papers to read:

1. "The Fundamental Theorem of Linear Algebra" [2]
2. "Too Much Calculus" [3]

Office hours:

- 12:50 pm to 3:50 pm, Wednesday,
  Farrell Hall, second floor, Trinity Campus

Grading breakdown:

1. Assignments (40%)
   - Ten one-week assignments.
   - Lowest assignment score will be dropped.
   - The last assignment cannot be dropped!
   - Each assignment will have a random bonus point question which has nothing to do with linear algebra.
2. Midterm exams (35%)
   - Three 75 minutes tests distributed throughout the course, all of equal weighting.
3. Final exam (24%)
   - $\leq$ Three hours of joyful celebration.
   - Monday, December 12, 1:30 pm to 4:15 pm, 254 Votey

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Grading breakdown:

4. Homework (0%)—Problems assigned online from the textbook. Doing these exercises will be most beneficial and will increase happiness.
5. General attendance (1%)—it is extremely desirable that students attend class, and class presence will be taken into account if a grade is borderline.

Questions are worth 3 points according to the following scale:

- 3 = correct or very nearly so.
- 2 = acceptable but needs some revisions.
- 1 = needs major revisions.
- 0 = way off.

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Schedule:

The course will mainly cover chapters 2 through 6 of the textbook. (You should know all about Chapter 1.)

Week # (dates)	Tuesday	Thursday
1 (8/30, 9/1)	Lecture	Lecture + A1
2 (9/6, 9/8)	Lecture	Lecture + A2
3 (9/13, 9/15)	Lecture	Lecture + A3
4 (9/20, 9/22)	Lecture	*Test 1*
5 (9/27, 9/29)	Lecture	Lecture + A4
6 (10/4, 10/6)	Lecture	Lecture + A5
7 (10/11, 10/13)	Lecture	Lecture + A6
8 (10/18, 10/20)	Lecture	*Test 2*
9 (10/25, 10/27)	Lecture	Lecture + A7
10 (11/1, 11/3)	Lecture	Lecture + A8
11 (11/8, 11/10)	Lecture	Lecture + A9
12 (11/15, 11/17)	Lecture	*Test 3*
13 (11/22, 11/24)	Thanksgiving	Thanksgiving
14 (11/29, 12/1)	Lecture + A10	Lecture
15 (12/6)	Lecture	—

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Important dates:

1. Classes run from Monday, August 29 to Wednesday, December 7.
2. Add/Drop, Audit, Pass/No Pass deadline—Monday, September 12.
3. Last day to withdraw—Monday, October 31 (Boo).
4. Reading and Exam period—Thursday, December 8 to Friday, December 16.

More stuff:

Do check your zoo account for updates regarding the course.

Academic assistance: Anyone who requires assistance in any way (as per the ACCESS program or due to athletic endeavors), please see or contact me as soon as possible.

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More stuff:

Being good people:

1. In class there will be no electronic gadgetry, no cell phones, no beeping, no text messaging, etc. You really just need your brain, some paper, and a writing implement here (okay, and Matlab or similar).
2. Second, I encourage you to email me questions, ideas, comments, etc., about the class but request that you please do so in a respectful fashion.
3. Finally, as in all UVM classes, Academic honesty will be expected and departures will be dealt with appropriately. See `http://www.uvm.edu/cses/` for guidelines.

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Even more stuff:

Late policy: Unless in the case of an emergency (a real one) or if an absence has been predeclared and a make-up version sorted out, assignments that are not turned in on time or tests that are not attended will be given 0%.

Computing: Students are encouraged to use Matlab or something similar to check their work.

Note: for assignment problems, written details of calculations will be required.

Why are we doing this?

Big deal: Linear Algebra is a body of mathematics that deals with discrete problems.

Many things are discrete:

- Information (0's & 1's, letters, words)
- People (sociology)
- Networks (the Web, people again, food webs, ...)
- Sounds (musical notes)

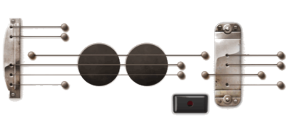
Even more:

If real data is continuous, we almost always discretize it (0's and 1's)

Why are we doing this?

Linear Algebra is used in many fields to solve problems:

- Engineering
- Computer Science
- Physics
- Economics
- Biology
- Ecology ...



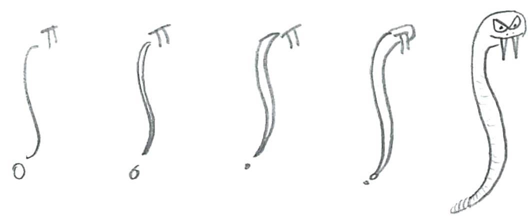
Big example: Google's Pagerank (⊞)

Some truth:

- Linear Algebra is as important as Calculus. . .
- Calculus ≡ the blue pill. . .

You are now choosing the red pill:

The Truth:



- Calculus is the Serpent's Mathematics.

The Platypus of Truth:

- Platypuses are masters of Linear Algebra.

The Truth:

Linear Algebra:

- Ghandi
- Buffy Summers
- Maple trees
- Chipmunks
- Elephants
- Yoda
- Hermione
- Frodo
- Indiana Jones
- Apple

Calculus:

- Poisonous spiders and other nasty bitey things
- Voldemort
- Big Bads
- Golem
- George Lucas
- Snakes
- Microsoft

Matrices as gadgets:

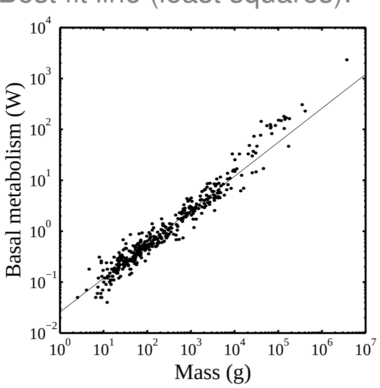
A matrix A transforms a vector $\vec{x}$ into a new vector $\vec{x}'$ through matrix multiplication (whatever that is):

$$\vec{x}' = A\vec{x}$$

We can use matrices to:

- Grow vectors
- Shrink vectors
- Rotate vectors
- Flip vectors
- Do all these things in different directions
- Reveal the true ur-dystopian reality.

Best fit line (least squares):

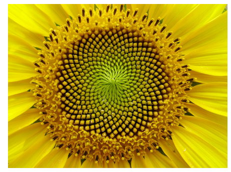


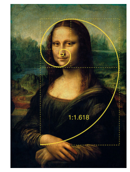
- Linear algebra does this beautifully;
- Calculus version is clunky. And evil.

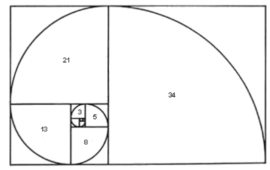
- From "Re-examination of the '3/4' law of metabolism"[1] Dodds, Rothman, and Weitz, Journal of Theoretical Biology, 209, 9–27, 2001

The many delights of Eigenthings:

Using Linear Algebra we'll somehow connect:

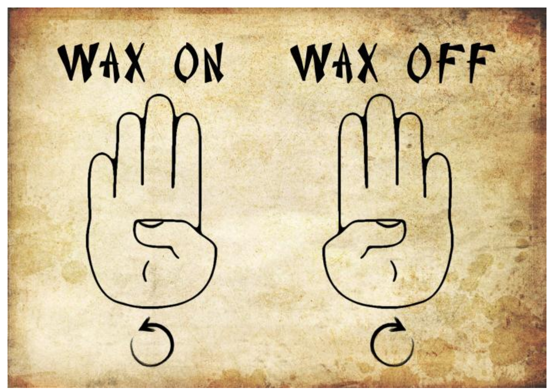






- Fibonacci Numbers,
- Golden Ratio,
- Spirals,
- Sunflowers, pine cones, ...
- Harvard Square.

This is a math course:



http://www.pimpartworks.com/artwork/randomsteveo/Wax-On-Wax-Off

- It's all connected. "More later."

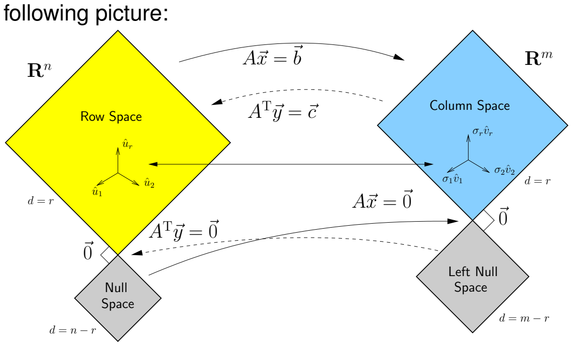
Three key problems of Linear Algebra

1. Given a matrix A and a vector $\vec{b}$, find $\vec{x}$ such that
$$A\vec{x} = \vec{b}.$$
2. Eigenvalue problem: Given A, find λ and $\vec{v}$ such that
$$A\vec{v} = \lambda\vec{v}.$$
3. Coupled linear differential equations:
$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = A\,y(t)$$

- Our focus will be largely on #1, partly on #2.

Major course objective:

To deeply understand the equation $A\vec{x} = \vec{b}$, the Fundamental Theorem of Linear Algebra, and the following picture:



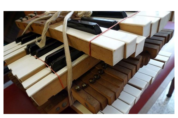
What is going on here? We have ~~25~~ 24 lectures to find out...

Our new BFF: $A\vec{x} = \vec{b}$

Broadly speaking, $A\vec{x} = \vec{b}$ translates as follows:

- $\vec{b}$ represents reality (e.g., music, structure)
- A contains building blocks (e.g., notes, shapes)
- $\vec{x}$ specifies how we combine our building blocks to make $\vec{b}$ (as best we can).

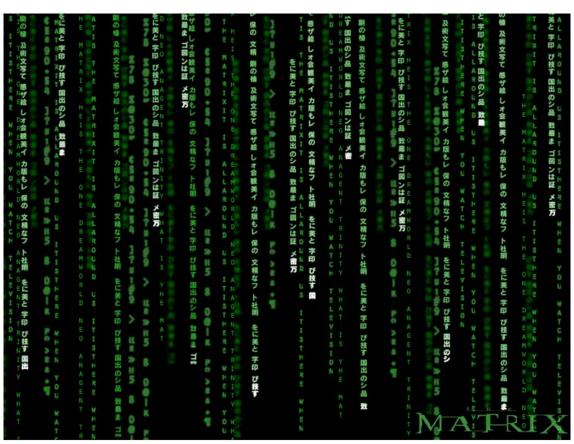
How can we disentangle an orchestra's sound?



- Radiolab (⊞)'s amazing piece: A 4-Track Mind (⊞)

What about pictures, waves, signals, ...?

Is this your left nullspace?:



Linear Algebra compliments/putdowns for Thanksgiving dinner:

- Wow, you have such a tiny/huge [delete as applicable] left nullspace!
- See also: The Dunning-Kruger effect. (⊞)

Our friend $A\vec{x} = \vec{b}$

What does knowing $\vec{x}$ give us?

If we can represent reality as a superposition (or combination or sum) of simple elements, we can do many things:

- Compress information
- See how we can alter information (filtering)
- Find a system's simplest representation
- Find a system's most important elements
- See how to adjust a system in a principled way

Three ways to understand $A\vec{x} = \vec{b}$:

- Way 1: The Row Picture
- Way 2: The Column Picture
- Way 3: The Matrix Picture

Example:

$$\begin{array}{rcrcl} -x_1 & + & x_2 & = & 1 \\ 2x_1 & + & x_2 & = & 4 \end{array}$$

- Call this a 2 by 2 system of equations.
- 2 equations with 2 unknowns.
- Standard method of simultaneous equations: solve above by adding and subtracting multiples of equations to each other = Row Picture.

Three ways to understand $A\vec{x} = \vec{b}$:

Row Picture—what we are doing:

- (a) Finding intersection of two lines
- (b) Finding the values of x_1 and x_2 for which both equations are satisfied (true/happy)
- A splendid and deep connection:
  (a) Geometry $\rightleftharpoons$ (b) Algebra

Three possible kinds of solution:

1. Lines intersect at one point —One, unique solution
2. Lines are parallel and disjoint —No solutions
3. Lines are the same —Infinitely many solutions

Three ways to understand $A\vec{x} = \vec{b}$:

The column picture:

See

$$\begin{array}{rcrcr} -x_1 & + & x_2 & = & 1 \\ 2x_1 & + & x_2 & = & 4 \end{array}$$

as

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

General problem

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$$

- Column vectors are our 'building blocks'
- Key idea: try to 'reach' $\vec{b}$ by combining (summing) multiples of column vectors $\vec{a}_1$ and $\vec{a}_2$.

Three ways to understand $A\vec{x} = \vec{b}$:

We love the column picture:

- Intuitive.
- Generalizes easily to many dimensions.

Three possible kinds of solution:

1. $\vec{a}_1 \not\parallel \vec{a}_2$: 1 solution
2. $\vec{a}_1 \parallel \vec{a}_2 \not\parallel \vec{b}$: No solutions
3. $\vec{a}_1 \parallel \vec{a}_2 \parallel \vec{b}$: infinitely many solutions

(assuming neither $\vec{a}_1$ or $\vec{a}_1$ are $\vec{0}$)

Three ways to understand $A\vec{x} = \vec{b}$:

Difficulties:

- Do we give up if $A\vec{x} = \vec{b}$ has no solution?
- No! We can still find the $\vec{x}$ that gets us as close to $\vec{b}$ as possible.
- Method of approximation—very important!
- We may not have the right building blocks but we can do our best.

Three ways to understand $A\vec{x} = \vec{b}$:

The Matrix Picture:

Now see

$$x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

as

$$A\vec{x} = \vec{b} : \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

A is now an operator:

- A transforms $\vec{x}$ into $\vec{b}$.
- Roughly speaking, A does two things to $\vec{x}$:
  1. Rotation/Flipping
  2. Dilation (stretching/contraction)

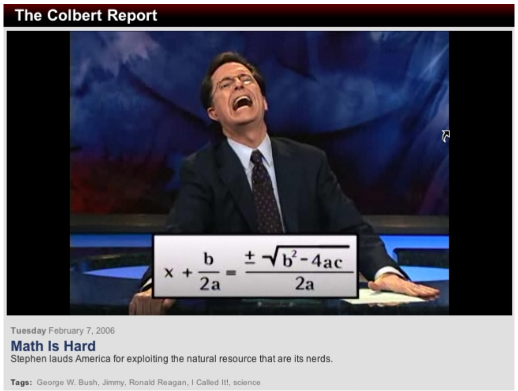
The Matrix Picture

Key idea in linear algebra:

- Decomposition or factorization of matrices.
- Matrices can often be written as products or sums of simpler matrices
- $A = LU$, $A = QR$, $A = U\Sigma V^{\mathrm{T}}$, $A = \sum_i \lambda_i \vec{v}\vec{v}^{\mathrm{T}}$, ...

More Truth about Mathematics:

The Colbert Report on Math (⊞) (February 7, 2006)



"Equations are the Devil's sentences."

References I

[1] P. S. Dodds, D. H. Rothman, and J. S. Weitz.
Re-examination of the "3/4-law" of metabolism.
Journal of Theoretical Biology, 209:9–27, 2001.
pdf (⊞)

[2] G. Strang.
The fundamental theorem of linear algebra.
The American Mathematical Monthly, 100(9):848–855, 1993. pdf (⊞)

[3] G. Strang.
Too much calculus, 2002.
SIAM Linear Algebra Activity Group Newsletter.
pdf (⊞)

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