Optimal Supply Networks

Complex Networks CSYS/MATH 303, Spring, 2011

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Optimal supply networks

What's the best way to distribute stuff?

- Stuff = medical services, energy people;
- ➤ Some fundamental network problems
 - 1. Distribute stuff from a single source to many sinks
 - 2. Distribute stuff from many sources to many sinks
 - 3. Redistribute stuff between nodes that are both sources and sinks
- Supply and Collection are equivalent problems

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Basic Q for distribution/supply networks:

How does flow behave given cost:

$$C = \sum_{j} I_{j}^{\gamma} Z_{j}$$

where

 I_i = current on link jand

 $Z_i = \text{link } j$'s impedance?

Basic Q for distribution/supply networks:

How does flow behave given cost:

$$C = \sum_{j} I_{j}^{\gamma} Z_{j}$$

where

 l_j = current on link j and

 $Z_i = \text{link } j$'s impedance?

Example: $\gamma = 2$ for electrical networks.

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(a) $\gamma > 1$: Braided (bulk) flow

(b) γ < 1: Local minimum: Branching flow

(c) γ < 1: Global minimum: Branching flow

From Bohn and Magnasco [3] See also Banavar et al. [1]

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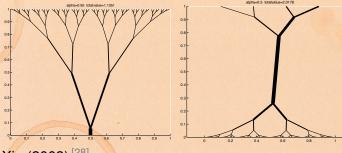
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Optimal paths related to transport (Monge) problems:



Xia (2003) [28]

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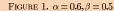
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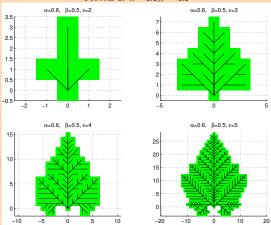
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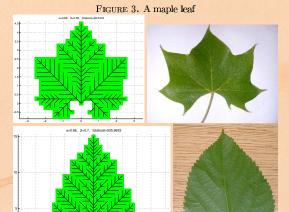
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Xia (2007) [27]





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An immensely controversial issue...

The form of river networks and blood networks: optimal or not? [26, 2, 5, 4]

Two observations:

- Self-similar networks appear everywhere in nature for single source supply/single sink collection.
- Real networks differ in details of scaling but reasonably agree in scaling relations.

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River network models

Optimality:

- Optimal channel networks [16]
- ► Thermodynamic analogy [17]

Randomness:

- Scheidegger's directed random networks
- Undirected random networks

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Optimality:

- Optimal channel networks [16]
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versus...

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Cardiovascular networks:

Murray's law (1926) connects branch radii at forks: [14, 13, 15, 10, 22]

$$r_0^3 = r_1^3 + r_2^3$$

where r_0 = radius of main branch and r_1 and r_2 are radii of sub-branches.

 See D'Arcy Thompson's "On Growth and Form" for background inspiration [21, 22].

Poiseuille flow (\boxplus)

[15, 11, 12]

▶ Use hydrautic equivalent of Ohm's law:

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- Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

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Cardiovascular networks:

Fluid mechanics: Poiseuille impedance (\boxplus) for smooth flow in a tube of radius r and length ℓ :

$$Z = \frac{8\eta\ell}{\pi r^4}$$

where $\eta = \text{dynamic viscosity } (\boxplus) \text{ (units: } ML^{-1}T^{-1}\text{)}.$

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Power required to overcome impedance:

$$P_{\text{drag}} = \Phi \Delta p = \Phi^2 Z$$
.

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▶ Power required to overcome impedance:

$$P_{\rm drag} = \Phi \Delta p = \Phi^2 Z$$
.

Also have rate of energy expenditure in maintaining blood:

$$P_{\rm metabolic} = cr^2 \ell$$

where *c* is a metabolic constant.

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Aside on P_{drag}

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Aside on P_{drag}

Work done = $F \cdot d$ = energy transferred by force F

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Aside on P_{drag}

- Work done = $F \cdot d$ = energy transferred by force F
- Power = P = rate work is done = $F \cdot v$

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- Work done = $F \cdot d$ = energy transferred by force F
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- $\triangleright \Delta p$ = Force per unit area

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- Work done = $F \cdot d$ = energy transferred by force F
- Power = P = rate work is done = $F \cdot v$
- $\triangleright \Delta p$ = Force per unit area
- Φ = Volume per unit time = cross-sectional area · velocity

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- Work done = $F \cdot d$ = energy transferred by force F
- Power = P = rate work is done = $F \cdot v$
- $\triangleright \Delta p$ = Force per unit area
- Φ = Volume per unit time
 = cross-sectional area · velocity
- So $\Phi \Delta p$ = Force · velocity

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► Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}}$$

- $\Phi^2 \frac{8\eta \ell}{\pi r^4} + cr^2 \ell$
- Observe power increases linearly with
- But is effect is nonlinear:
 - increasing r makes flow easier but increases metabolic cost (as r²)
 - ► decreasing r decrease metabolic cost but impedance goes up (as r^{-4})

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Minimize P with respect to r:

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r} \left(\Phi^2 \frac{8\eta \ell}{\pi r^4} + cr^2 \ell \right)$$

 $-4\Phi^2 \frac{8\eta\ell}{\pi r^6} + c2r\ell = 0$

► Rearrange/cancel/slap:

$$\Phi^2 = \frac{c\pi r^6}{16n} = k^2 r^6$$

where k = constant.

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So we now have:

$$\Phi = kr^3$$

Flow rates at each branching have to add up (else our organism is in serious trouble...):

 $\Phi_0 = \Phi_1 + \Phi_2$

where again 0 refers to the main branch and 1 and 2

All of this means we have a groovy cribe-lay

 $r_0^3 = r_1^3 + r_2^3$

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where again 0 refers to the main branch and 1 and 2 refers to the offspring branches

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$$r_0^3 = r_1^3 + r_2^3$$

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 $lackbox{\Phi}_{\omega}$ = volume rate of flow into an order ω vessel segment

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Murray meets Tokunaga:

- $Φ_ω$ = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

$$\Phi_{\omega} = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

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- Φ_{ω} = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

$$\Phi_{\omega} = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

• Using $\phi_{\omega} = kr_{\omega}^3$

$$r_{\omega}^{3} = 2r_{\omega-1}^{3} + \sum_{k=1}^{\omega-1} T_{k} r_{\omega-k}^{3}$$

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Murray meets Tokunaga:

- $Φ_ω$ = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

$$\Phi_{\omega} = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

• Using $\phi_{\omega} = kr_{\omega}^3$

$$r_{\omega}^{3} = 2r_{\omega-1}^{3} + \sum_{k=1}^{\omega-1} T_{k} r_{\omega-k}^{3}$$

Find Horton ratio for vessel radius $R_r = r_{\omega}/r_{\omega-1}...$

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Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

$$R_r^3 = R_n = R_v$$

> Is there more we could do here to constrain the

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Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

$$R_r^3 = R_n = R_v$$

Is there more we could do here to constrain the Horton ratios and Tokunaga constants?

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▶ Isometry: $V_{\omega} \propto \ell_{\omega}^3$

 $R_{\ell}^3 = R_{\nu} = R_n$

- ➤ We need one more constraint.
- ➤ West et al (1997) [26] achieve similar results following

So does Turcotte et al. (1998) [23] using Tokunaga

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Murray meets Tokunaga:

- Isometry: $V_{\omega} \propto \ell_{\omega}^3$
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$$R_{\ell}^3 = R_v = R_n$$

- We need one more constraint...
- ▶ West et al (1997) [26] achieve similar results following Horton's laws.
- So does Turcotte et al. (1998) [23] using Tokunaga (sort of).

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D-----



Geometric argument

- Consider one source supplying many sinks in a volume V d-dim. region in a D-dim. ambient space.
- Assume sinks are invariant.
- Assume $\rho = \rho(V)$, i.e., ρ may vary with region's volume V.

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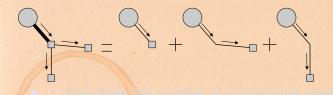
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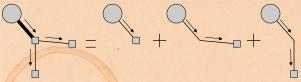
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Supply Networks

Geometric argument

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- See network as a bundle of virtual vessels:



Q: how does the number of sustainable sinks N_{sinks} scale with volume V for the most efficient network design?

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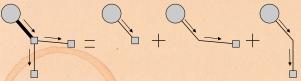
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Supply Networks

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- Assume $\rho = \rho(V)$, i.e., ρ may vary with region's volume V.
- See network as a bundle of virtual vessels:



- Q: how does the number of sustainable sinks N_{sinks} scale with volume V for the most efficient network design?
- ▶ Or: what is the highest α for $N_{\text{sinks}} \propto V^{\alpha}$?

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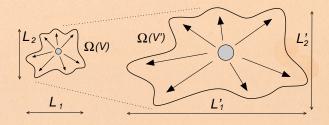
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Geometric argument

Allometrically growing regions:



Have d length scales which scale as

$$L_i \propto V^{\gamma_i}$$
 where $\gamma_1 + \gamma_2 + \ldots + \gamma_d = 1$.

- For isometric growth, $\gamma_i = 1/d$.
- For allometric growth, we must have at least two of the $\{\gamma_i\}$ being different

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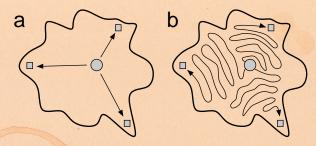
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▶ Best and worst configurations (Banavar et al.)



► Rather obviously:

min V. S. S. distances from source to sinks

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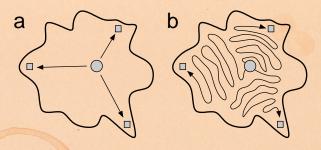
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▶ Best and worst configurations (Banavar et al.)



► Rather obviously: min $V_{\text{net}} \propto \sum$ distances from source to sinks.

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Minimal network volume:

Real supply networks are close to optimal:

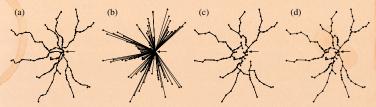


Figure 1. (a) Commuter rail network in the Boston area. The arrow marks the assumed root of the network. (b) Star graph. (c) Minimum spanning tree. (d) The model of equation (3) applied to the same set of stations.

(2006) Gastner and Newman [8]: "Shape and efficiency in spatial distribution networks"

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Minimal network volume:

Add one more element:

- Vessel cross-sectional area may vary with distance from the source.
- Flow rate increases as cross-sectional area decreases.
- e.g., a collection network may have vessels tapering as they approach the central sink.
- Find that vessel volume v must scale with vessel length ℓ to affect overall system scalings.
- Consider vessel radius $r \propto (\ell + 1)^{-\epsilon}$, tapering from $r = r_{\text{max}}$ where $\epsilon \geq 0$.
- Gives $v \propto \ell^{1-2\epsilon}$ if $\epsilon < 1/2$
- ▶ Gives $\nu \propto 1 \ell^{-(2\epsilon-1)} \rightarrow 1$ for large ℓ if $\epsilon > 1/2$
- Previously, we looked at $\epsilon = 0$ only.

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For $0 \le \epsilon < 1/2$, approximate network volume by integral over region:

$$\min V_{\rm net} \propto \int_{\Omega_{d,D}(V)} \rho \, ||\vec{x}||^{1-2\epsilon} \, \mathrm{d}\vec{x}$$

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For $0 \le \epsilon < 1/2$, approximate network volume by integral over region:

$$egin{aligned} \mathsf{min} \ V_{\mathsf{net}} & \propto \int_{\Omega_{d,D}(V)}
ho \left| \left| ec{x}
ight|
ight|^{1-2\epsilon} \mathrm{d}ec{x} \end{aligned}$$

Insert question 1, assignment 3 (⊞)

$$\propto
ho V^{1+\gamma_{\max}(1-2\epsilon)}$$
 where $\gamma_{\max} = \max_i \gamma_i$.

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Insert question 1, assignment 3 (⊞)

$$\propto \rho V^{1+\gamma_{\max}(1-2\epsilon)}$$
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For $\epsilon > 1/2$, find simply that

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$$V_{\rm net} \propto \rho V$$

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Minimal network volume:

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 where $\gamma_{\max} = \max_{i} \gamma_{i}$.

For $\epsilon > 1/2$, find simply that

min
$$V_{\rm net} \propto \rho V$$

So if supply lines can taper fast enough and without limit, minimum network volume can be made negligible.

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For $0 \le \epsilon < 1/2$:

- ► If scaling is isometric we have $\gamma_{\text{miax}} = 1/d$

 $\min V_{\text{nor}} t_{\text{iso}} \propto \rho V^{1+(1-2\epsilon)}$

It scaling is allometric, we have $\gamma_{max} = \gamma_{allo} > 1/d$: and

min $V_{\rm ret ballo} \propto \rho V^{1+(1-2\epsilon)}$

sometrically growing volumes require less network volume than allometrically growing volumes:

min V_{ner/silo} — 0 as V — ∞

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For $0 \le \epsilon < 1/2$:

- If scaling is isometric, we have $\gamma_{max} = 1/d$:

$$\min V_{\text{net/iso}} \propto \rho V^{1+(1-2\epsilon)/d}$$

- It soaling is allometric, we have $\gamma_{\text{max}} = \gamma_{\text{ullo}} > 1/d$: and
- sometrically growing volumes require less network

min $V_{
m nevalur} = 0$ as $V = \infty$

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Isometrically growing volumes require less network volume than allometrically growing volumes:

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ightarrow \infty$$

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For $\epsilon > 1/2$:

- Network volume scaling is now independent of overall shape scaling.

Limits to scaling

- Can argue that a must effectively be 0 for real networks over large enough scales.
- Limit to how fast material can move, and how small material packages can be.
- ▶ e.g., blood velocity and blood cell size

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- Velocity at capillaries and aorta approximately constant across body size [25]: $\epsilon = 0$.
- ► Material costly ⇒ expect lower optimal bound of
- For cardiovascular networks, d = D = 3
- ► Blood volume scales linearly with blood volume [18]
- ➤ Sink density must decrease as volume increases:

 $\rho \propto V^{-1/d}$

Density of suppliable sinks decreases with organism

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Then P, the rate of overall energy use in Ω, can at most scale with volume as

$$P \propto \rho V$$

For d = 3 dimensional organisms, we have

$$P \propto M^{2/3}$$

- Including other constraints may raise scaling exponent to a higher less efficient value.
- ► Exciting bonus: Scaling obtained by the supply network story and the surface area law only match for isometrically growing shapes. Insert question 3. assignment 3 (⊞)

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Recap:

- The exponent $\alpha = 2/3$ works for all birds and mammals up to 10–30 kg
- For mammals > 10–30 kg, maybe we have a new scaling regime
- ► Economos: limb length break in scaling around 20 kg
- White and Seymour, 2005: unhappy with large
- herbivore measurements. Find $a \simeq 0.686 \pm 0.014$

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- View river networks as collection networks.
- Many sources and one sink.
- Assume ρ is constant over time and $\epsilon = 0$:
 - $V_{\rm net} \propto \rho V^{(d+1)/d} = {\rm constant} \times V^3$
- Network volume grows faster than basin 'volume'
- ► It's all okay:
- Landscapes are d=2 surfaces living in D=3
- > Streams can grow not just in width but in depth
- ▶ If a > 0. V_{no} will grow more slowly but 3/2 appears to
 - be confirmed from real data.

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- View river networks as collection networks.
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How do we distribute sources?

- Focus on 2-d (results generalize to highe dimensions)
- ➤ Sources = hospitals, post offices, pubs
- Key problem: How do we cope with uneven population densities?
- Obvious: it density is uniform then sources are best distributed uniformly
- Which lattice is optimal? The hexagonal lattice Q1: How big should the hexagons be?
- Q2: Given population density is uneven, what do we do?
- ► We is follow work by Stephan^[19, 20], Gastner and Newman (2006) ^[7]. Um *et al.* ^[24] and work cited by

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Solidifying the basic problem

- Given a region with some population distribution ρ , most likely uneven.
- Given resources to build and maintain N facilities
- ► Q: How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?

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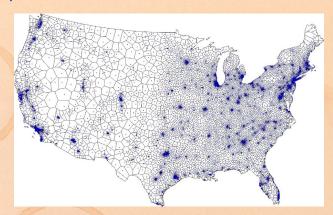
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From Gastner and Newman (2006) [7]

- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.
- Simulated annealing + Voronoi tessellation.

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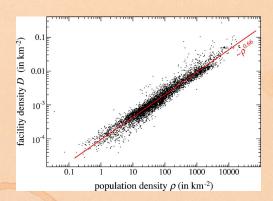
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From Gastner and Newman (2006) [7]

▶ Optimal facility density D vs. population density ρ .

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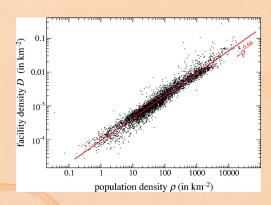
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From Gastner and Newman (2006) [7]

- ▶ Optimal facility density D vs. population density ρ .
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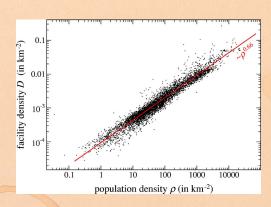
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From Gastner and Newman (2006) [7]

- ▶ Optimal facility density D vs. population density ρ .
- Fit is $D \propto \rho^{0.66}$ with $r^2 = 0.94$.
- Looking good for a 2/3 power...

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 $D \propto
ho^{2/3}$

- > Why
- Again: Different story to branching networks where there was either one source or one sink.
- Now sources & sinks are distributed throughout region...

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- ► We first examine Stephan's treatment (1977) [19, 20]
- "Territorial Division: The Least-Time Constraint Behind the Formation of Subnational Boundaries" (Science, 1977)
- Zipf-like approach: invokes principle of minimal effort
- Also known as the Homer principle

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- Consider a region of area A and population P with a single functional center that everyone needs to access every day.
- Build up a general cost function based on time expended to access and maintain center.
- Write average travel distance to center as d and assume average speed of travel is v.
- Assume isometry: average travel distance \tilde{d} will be on the length scale of the region which is $\sim A^{1/2}$
- Average time expended per person in accessing facility is therefore

$$\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$$

where c is an unimportant shape factor.

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 Next assume facility requires regular maintenance (person-hours per day)

- Call this quantity
- If burden of mainenance is shared then average copper person is τ/P where P = population.
- ▶ Replace P by ρA where ρ is density.
- ▶ Total average time cost per person

$$/\bar{\mathbf{v}} + \tau/(\rho \mathbf{A}) = g\mathbf{A}^{1/2}/\bar{\mathbf{v}} + \tau/(\rho \mathbf{A})$$

▶ Now Minimize with respect to A...

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- ▶ Call this quantity τ
- If burden of mainenance is shared then average cost per person is τ/P where P = population.
- ▶ Replace P by A where is density.
- ▶ Total average time cost per person

$$q = qA^{1/2}/\bar{v} + \tau/(\rho A)$$

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$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A} \left(c A^{1/2} / \bar{v} + \tau / (\rho A) \right)$$

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$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A} \left(cA^{1/2} / \bar{v} + \tau / (\rho A) \right)$$
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► Rearrange:

$$A = \left(\frac{2\bar{v}\tau}{c\rho}\right)^{2/3}$$

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ho}
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▶ # facilities per unit area

$$A^{-1} \propto \rho^{2/3}$$

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ight)^{2/3} \propto
ho^{-2/3}$$

▶ # facilities per unit area

$$A^{-1} \propto \rho^{2/3}$$

► Groovy...

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An issue:

 Maintenance (τ) is assumed to be independent of population and area (P and A)

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Stephan's online book "The Division of Territory in Society" is here (⊞).

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Standard world map:



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Cartogram of countries 'rescaled' by population:



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Diffusion-based cartograms:

$$\nabla^2 \rho - \frac{\partial \rho}{\partial t} = 0$$

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Diffusion-based cartograms:

- Idea of cartograms is to distort areas to more accurately represent some local density ρ (e.g. population).
- Many methods put forward—typically involve some kind of physical analogy to spreading or repulsion.
- ➤ Algorithm due to Gastner and Newman (2004) ^[6] is based on standard diffusion:

- Allow density to diffuse and trace the movement of
- ➤ Diffusion is constrained by boundary condition of surrounding area having density 5.

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Allow density to diffuse and trace the movement of individual elements and boundaries.

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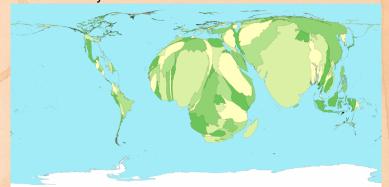
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Child mortality:



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Energy consumption:



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Gross domestic product:



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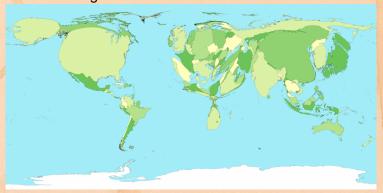
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Greenhouse gas emissions:



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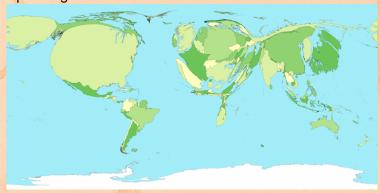
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Spending on healthcare:



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People living with HIV:



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The preceding sampling of Gastner & Newman's cartograms lives here (⊞).

 A larger collection can be found at worldmapper.org (H)



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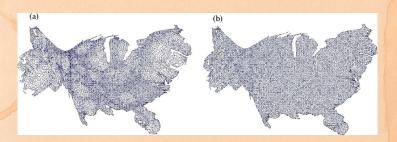
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Size-density law



Left: population density-equalized cartogram.

Right: (population density)2/8-equalized cartogra

From Gastner and Newman (2006) [7]

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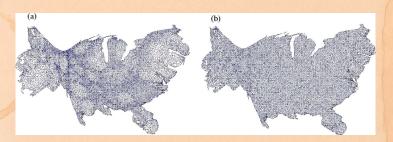
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Size-density law



- Left: population density-equalized cartogram.
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From Gastner and Newman (2006) [7]

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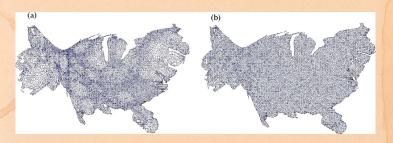
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- Left: population density-equalized cartogram.
- Right: (population density)^{2/3}-equalized cartogram.
- Facility density is uniform for $\rho^{2/3}$ cartogram.

From Gastner and Newman (2006) [7]

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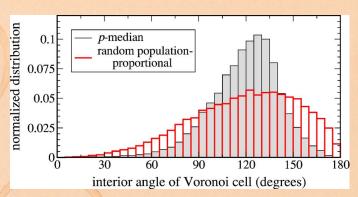
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From Gastner and Newman (2006) [7]

Cartogram's Voronoi cells are somewhat hexagonal.

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Size-density law

Deriving the optimal source distribution:

$$F(\{\vec{x}_1, \dots, \vec{x}_n\}) = \int_{\Omega} \rho(\vec{x}) \min_{i} ||\vec{x} - \vec{x}_i|| dx$$

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Deriving the optimal source distribution:

Basic idea: Minimize the average distance from a random individual to the nearest facility. [7]

Assume given a fixed population density ρ defined of a spatial radius Ω

spatial region 12.

Formally, we want to find the locations of *n* sources that minimizes the cost function

 $\{\vec{x}_n\}$ = $\{\rho(\vec{x}) \min ||\vec{x} - \vec{x}_i|| d\vec{x}\}$

Also known as the p-median problem

Not easy... in fact this one is an NP-hard problem. [7]

Approximate solution originally due to

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$$F(\{\vec{x}_1,\ldots,\vec{x}_n\}) = \int_{\Omega} \rho(\vec{x}) \min_i ||\vec{x} - \vec{x}_i|| d\vec{x}.$$

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- Also known as the p-median problem.
- ▶ Not easy... in fact this one is an NP-hard problem. [7]
- Approximate solution originally due to Gusein-Zade [9].

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Approximations:

- For a given set of source placements $\{\vec{x}_1, \dots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells (\boxplus) , one per source.
- \blacktriangleright Define A(x) as the area of the Voronoi cell containing
- ▶ As per Stephan's calculation, estimate typical distance from x to the nearest source (say i) as

 $c_i A(\vec{x})^{1/2}$

shape factor for the ith Voronoi cell

► Approximate c: as a constant c

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Carrying on:

The cost function is now

$$F = c \int_{\Omega} \rho(\vec{x}) A(\vec{x})^{1/2} d\vec{x}.$$

- ▶ We also have that the constraint that Voronoi cells divide up the overall area of Ω: $\sum_{i=1}^{n} A(\vec{x_i}) = A_{\Omega}$.
- Sneakily turn this into an integral constraint:

 $\int_{\mathcal{O}} \frac{\mathrm{d}X}{A(\vec{X})} = I$

- ightharpoonup Within each cell $A(\bar{x})$ is constant
- ➤ So. integral over each of the n cells equals 1

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So integral over each of the a cells equals

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- ▶ Within each cell, $A(\vec{x})$ is constant.
- So... integral over each of the *n* cells equals 1.

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Now a Lagrange multiplier story:

▶ By varying $\{\vec{x}_1, ..., \vec{x}_n\}$, minimize

$$G(A) = c \int_{\Omega} \rho(\vec{x}) A(\vec{x})^{1/2} d\vec{x} - \lambda \left(n - \int_{\Omega} \left[A(\vec{x}) \right]^{-1} d\vec{x} \right)$$

- Next compute $\delta G/\delta A$, the functional derivative (\boxplus) the functional G(A).
- This gives

 $\left[\rho(\vec{x})A(\vec{x})^{-1/2} - \lambda \left[A(\vec{x}) \right]^{-2} \right] \mathrm{d}\vec{x} = 0$

> Setting the integrand to be zilch, we have

 $\rho(\vec{x}) = 2\lambda c^{-1} A(\vec{x})^{-3/2}$

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Now a Lagrange multiplier story:

Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho^{-2/3}.$$

Finally, we indentify $1/A(\vec{x})$ as $D(\vec{x})$, an approximation of the local source density

▶ Substituting D = 1/A, we have

$$D(\vec{x}) = \left(\frac{c}{2\lambda}\rho\right)^{2/3}$$

Normalizing for solving for A:

 $D(\vec{x}) = 0 \frac{|\rho(\vec{x})|^{2/3}}{|\rho(\vec{x})|^{2/3} d\vec{x}} \propto [\rho(\vec{x})]^{2/3}$

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Rearranging, we have

$$A(\vec{x}) = (2\lambda c^{-1})^{2/3} \rho^{-2/3}.$$

- Finally, we indentify $1/A(\vec{x})$ as $D(\vec{x})$, an approximation of the local source density.
- ▶ Substituting D = 1/A, we have

$$D(\vec{x}) = \left(\frac{c}{2\lambda}\rho\right)^{2/3}.$$

Normalizing (or solving for λ):

$$D(\vec{x}) = n \frac{[\rho(\vec{x})]^{2/3}}{\int_{\Omega} [\rho(\vec{x})]^{2/3} d\vec{x}} \propto [\rho(\vec{x})]^{2/3}.$$

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One more thing:

How do we supply these facilities?

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One more thing:

- How do we supply these facilities?
- How do we best redistribute mail? People?

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One more thing:

- How do we supply these facilities?
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- How do we get beer to the pubs?

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One more thing:

- How do we supply these facilities?
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- Gaster and Newman model: cost is a function of basic maintenance and travel time:

$$C_{\text{maint}} + \gamma C_{\text{travel}}$$
.

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Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance ℓ_{ij} and number of legs to journey:

$$(1-\delta)\ell_{ij}+\delta(\#\mathsf{hops}).$$

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▶ When $\delta = 1$, only number of hops matters.

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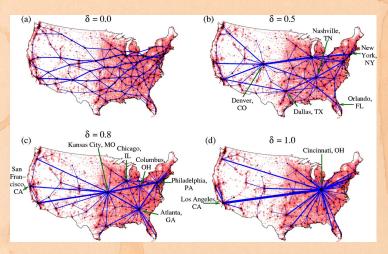
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From Gastner and Newman (2006) [7]

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Beyond minimizing distances:

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Beyond minimizing distances:

- "Scaling laws between population and facility densities" by Um et al., Proc. Natl. Acad. Sci., 2009. [24]
 - Um et al. find empirically and argue theoretically the connection between facility and population density

 $D \propto n^{\alpha}$

does not universally hold with $\alpha = 2/3$.

- Two idealized limiting classes
 - 1. For-profit, commercial facilities: $\alpha = 1$
 - 2. Pro-social, public facilities: $\alpha = 2/3$
- Um et al. investigate facility locations in the United States and South Korea.

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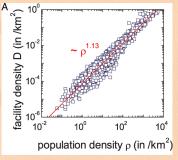
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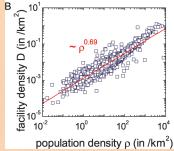
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Public versus private facilities: evidence





- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.

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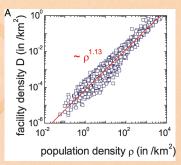
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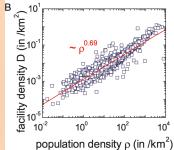
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- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.
- Note: break in scaling for public schools. Transition from $\alpha \simeq 2/3$ to $\alpha = 1$ around $\rho \simeq 100$.

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US facility	α (SE)	R ²
Ambulatory hospital	1.13(1)	0.93
Beauty care	1.08(1)	0.86
Laundry	1.05(1)	0.90
Automotive repair	0.99(1)	0.92
Private school	0.95(1)	0.82
Restaurant	0.93(1)	0.89
Accommodation	0.89(1)	0.70
Bank	0.88(1)	0.89
Gas station	0.86(1)	0.94
Death care	0.79(1)	0.80
* Fire station	0.78(3)	0.93
* Police station	0.71(6)	0.75
Public school	0.69(1)	0.87
SK facility	α (SE)	R ²
SK facility Bank	α (SE)	R ²
Bank	1.18(2)	0.96
Bank Parking place	1.18(2) 1.13(2)	0.96 0.91
Bank Parking place * Primary clinic	1.18(2) 1.13(2) 1.09(2)	0.96 0.91 1.00
Bank Parking place * Primary clinic * Hospital	1.18(2) 1.13(2) 1.09(2) 0.96(5)	0.96 0.91 1.00 0.97
Bank Parking place * Primary clinic * Hospital * University/college	1.18(2) 1.13(2) 1.09(2) 0.96(5) 0.93(9)	0.96 0.91 1.00 0.97 0.89
Bank Parking place * Primary clinic * Hospital * University/college Market place	1.18(2) 1.13(2) 1.09(2) 0.96(5) 0.93(9) 0.87(2)	0.96 0.91 1.00 0.97 0.89 0.90
Bank Parking place * Primary clinic * Hospital * University/college Market place * Secondary school	1.18(2) 1.13(2) 1.09(2) 0.96(5) 0.93(9) 0.87(2) 0.77(3)	0.96 0.91 1.00 0.97 0.89 0.90
Bank Parking place * Primary clinic * Hospital * University/college Market place * Secondary school * Primary school	1.18(2) 1.13(2) 1.09(2) 0.96(5) 0.93(9) 0.87(2) 0.77(3) 0.77(3)	0.96 0.91 1.00 0.97 0.89 0.90 0.98
Bank Parking place * Primary clinic * Hospital * University/college Market place * Secondary school * Primary school Social welfare org.	1.18(2) 1.13(2) 1.09(2) 0.96(5) 0.93(9) 0.87(2) 0.77(3) 0.77(3) 0.75(2)	0.96 0.91 1.00 0.97 0.89 0.90 0.98 0.97
Bank Parking place * Primary clinic * Hospital * University/college Market place * Secondary school * Primary school Social welfare org. * Police station	1.18(2) 1.13(2) 1.09(2) 0.96(5) 0.93(9) 0.87(2) 0.77(3) 0.77(3) 0.75(2) 0.71(5)	0.96 0.91 1.00 0.97 0.89 0.90 0.98 0.97 0.84 0.94

Rough transition between public and private at $\alpha \simeq 0.8$.

Note: * indicates analysis is at state/province level; otherwise county level.

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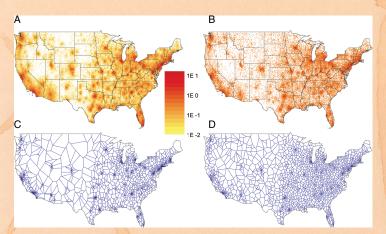
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Public versus private facilities: evidence



A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.

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So what's going on?

- Social institutions seek to minimize distance of travel.

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Public versus private facilities: the story So what's going on?

- Social institutions seek to minimize distance of travel.
- Commercial institutions seek to maximize the number of visitors.

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- ► Social institutions seek to minimize distance of travel.
- Commercial institutions seek to maximize the number of visitors.
- Defins: For the *i*th facility and its Voronoi cell V_i , define
 - $n_i =$ population of the *i*th cell;
 - $\langle r_i \rangle$ = the average travel distance to the *i*th facility.
 - $ightharpoonup s_i = area of ith cell.$

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Public versus private facilities: the story So what's going on?

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 - $ightharpoonup s_i = area of ith cell.$
- Objective function to maximize for a facility (highly constructed):

$$v_i = n_i \langle r_i \rangle^{\beta}$$
 with $0 \le \beta \le 1$.

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- Objective function to maximize for a facility (highly constructed):

$$v_i = n_i \langle r_i \rangle^{\beta}$$
 with $0 \le \beta \le 1$.

- Limits:
 - $\beta = 0$: purely commercial.
 - $\beta = 1$: purely social.

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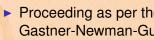
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Proceeding as per the Gastner-Newman-Gusein-Zade calculation, Um et al. obtain:

$$D(\vec{x}) = n \frac{[\rho(\vec{x})]^{2/(\beta+2)}}{\int_{\Omega} [\rho(\vec{x})]^{2/(\beta+2)} d\vec{x}} \propto [\rho(\vec{x})]^{2/(\beta+2)}.$$

Public versus private facilities: the story

Proceeding as per the Gastner-Newman-Gusein-Zade calculation, Um et al. obtain:

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▶ For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.

Lyo. Social scanny is submitted.

Insert question 3, assignment

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- ▶ For $\beta = 0$, $\alpha = 1$: commercial scaling is linear.
- For $\beta = 1$, $\alpha = 2/3$: social scaling is sublinear.
- You can try this too: Insert question 3, assignment 4 (⊞).

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