# Scale-Free Networks Complex Networks CSYS/MATH 303, Spring, 2011

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- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail

 $P_k \sim k^{-\gamma}$  for 'large' k

- One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999 "Emergence of scaling in random networks" [2]
- Somewhat misleading nomenclature

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- Scale-free networks are not fractal in any sense.



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- Usually talking about networks whose links are abstract, relational, informational, ... (non-physical)

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- Primary example: hyperlink network of the Web

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- Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, ... (non-physical)
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not...

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## Random networks: largest components









$$\gamma = 2.5$$
 $\langle k \rangle = 1.8$ 

 $\gamma = 2.5$  $\langle k \rangle = 2.05333$ 



$$\gamma = 2.5$$
 $\langle k \rangle = 1.92$ 









$$\gamma = 2.5$$
 $\langle k \rangle = 1.6$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.50667$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.62667$ 

 $\gamma = 2.5$  $\langle k \rangle = 1.8$ 

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## The big deal:

We move beyond describing of networks to finding mechanisms for why certain networks are the way they are.

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We move beyond describing of networks to finding mechanisms for why certain networks are the way they are.

## A big deal for scale-free networks:

• How does the exponent  $\gamma$  depend on the mechanism?

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## The big deal:

We move beyond describing of networks to finding mechanisms for why certain networks are the way they are.

## A big deal for scale-free networks:

- How does the exponent  $\gamma$  depend on the mechanism?
- Do the mechanism details matter?

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## Work that presaged scale-free networks

- ► 1924: G. Udny Yule [9]: # Species per Genus
- ▶ 1926: Lotka [4]
- # Scientific papers per author
- ▶ 1953: Mandelbrot [5]
  - Zipf's law for word frequency through optimization
- ▶ 1955: Herbert Simon [8, 10]
  - Zipt's law, city size, income, publications, and species per denus
  - species per genus
- ▶ 1965/1976: Derek de Solla Price [6, 7]
  - Network of Scientific Citations

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- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA)
- Step 1: start with m₀ disconnected nodes
- ► Step 2
  - 1. Growth—a new node appears at each time step  $t = 0, 1, 2, \dots$
  - 2. Each new node makes *m* links to nodes already present.
  - 3. Preferential attachment—Probability of connecting to ith node is  $\propto k_i$ .
- ▶ In essence, we have a rich-gets-richer scheme

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**Definition:**  $A_k$  is the attachment kernel for a node with degree k.

- Definition: P (It I) is the attachment probability
- For the original model

$$\frac{k_{i}(t)}{\sum_{k=1}^{k_{max}(t)} k_{j}(t)} = \frac{k_{i}(t)}{\sum_{k=m}^{k_{max}(t)} k N_{k}(t)}$$

where  $N(t) = m_0 + t$  is # nodes at time t and  $N_k(t)$  is # degree k nodes at time t.

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- **Definition:**  $A_k$  is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- ▶ Definition: P<sub>weeb</sub>(k, t) is the attachment probability
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- ▶ Definition:  $A_k$  is the attachment kernel for a node with degree k.
- For the original model:

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**Definition:**  $P_{\text{attach}}(k, t)$  is the attachment probability.

$$= \frac{K_i(t)}{\sum_{k=m}^{k_{\max}(t)} k N_k(t)}$$

where  $N(t) = m_0 + t$  is # nodes at time t and  $N_k(t)$  is # degree k nodes at time t.

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$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

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# BA model

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When (N + 1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1}-k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small
- Dispense with Expectation by assuming (hoping) that

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m \frac{k_i(t)}{\sum_{i=1}^{N(t)} k_i(t)}$$

where  $t \neq N(t) - m_0$ .

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# $\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m \frac{k_i(t)}{\sum_{i=1}^{N(t)} k_i(t)}$

where 
$$t = N(t) - m_0$$
.

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- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m \frac{k_i(t)}{\sum_{i=1}^{N(t)} k_i(t)}$$

where 
$$t = N(t) - m_0$$
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- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- Approximate  $k_{i,N+1} k_{i,N}$  with  $\frac{d}{dt}k_{i,t}$ :

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m \frac{k_i(t)}{\sum_{i=1}^{N(t)} k_i(t)}$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)}$$

where  $t = N(t) - m_0$ .

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Deal with denominator: each added node brings m new edges.

$$\frac{k_i(t)}{2mt} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$



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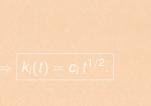




Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{i=1}^{N(t)} k_j(t) = 2tm$$

$$\frac{k_i(t)}{M(0)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$



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Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m \frac{k_i(t)}{\sum_{i=1}^{N(t)} k_i(t)}$$

$$k_i(t) = c_i t^{1/2}$$

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# $k_i(t) = c_i t^{1/2}$

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$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t}$$

#### Scale-Free Networks

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Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

The node degree equation now simplifies:

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▶ Next find c<sub>i</sub> ...

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NOTTIOIS







Know ith node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

▶ So for  $i > m_0$  (exclude initial nodes), we must have

All node decrees grow as  $t^{1/2}$  but later nodes have larger  $t_{i \text{ start}}$  which flattens out growth curve.

Larly nodes do best (Hirst-mover advantage)

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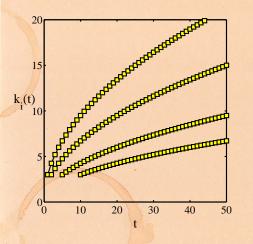
### Redner & Krapivisky's model

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- $\rightarrow m=3$
- $t_{i,\text{start}} = 1, 2, 5, \text{ and } 10.$

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- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distribute uniformly;
  - $P(t_{i,\text{stain}}) dt_{i,\text{stain}} \simeq \frac{dt_{i,\text{stain}}}{t + m_0}$

Using

$$k_i(t) = m\left(\frac{t}{t_{i,\text{start}}}\right)^{\sqrt{s}} \Rightarrow t_{i,\text{start}} = \frac{m \cdot t}{k_i(t)}$$

and by understanding that later arriving nodes have lower degrees, we can say this:

$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2})$$

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Using the uniformity of start times:

$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

▶ Differentiate to find Pr(k)

$$\operatorname{Pr}(k) = \frac{d}{dk}\operatorname{Pr}(k, < k) = \frac{2m^2t}{(t+m_0)k^3}$$

$$\sim 2m^2k^{-3}$$
 as  $m\to\infty$ .

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- We thus have a very specific prediction of  $Pr(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .

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- We thus have a very specific prediction of  $Pr(k) \sim k^{-\gamma}$  with  $\gamma = 3$ .
- Typical for real networks:  $2 < \gamma < 3$ .
- Range true more generally for events with size
- ▶ 2 < √ < 3. finite mean and 'infinite' variance (wild)
- In practice, 
   3 means variance is governed by upper cutoff

   γ > 3 finite mean and variance (mild)

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# Degree distribution

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# **Examples**

WWW WWW

 $\gamma \simeq$  2.1 for in-degree  $\gamma \simeq$  2.45 for out-degree

Movie actors  $\gamma \simeq$  2.3

 $\gamma \simeq 2.8$ Words (synonyms)

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# **Examples**

WWW  $\gamma \simeq$  2.1 for in-degree WWW  $\gamma \simeq$  2.45 for out-degree Movie actors  $\gamma \simeq 2.3$ Words (synonyms)  $\gamma \simeq 2.8$ 

The Internets is a different business...

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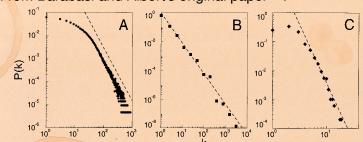






## Real data

## From Barabási and Albert's original paper [2]:



**Fig. 1.** The distribution function of connectivities for various large networks. **(A)** Actor collaboration graph with N=212,250 vertices and average connectivity  $\langle k \rangle=28.78$ . **(B)** WWW, N=325,729,  $\langle k \rangle=5.46$  **(6)**. **(C)** Power grid data, N=4941,  $\langle k \rangle=2.67$ . The dashed lines have slopes (A)  $\gamma_{\rm actor}=2.3$ , (B)  $\gamma_{\rm www}=2.1$  and (C)  $\gamma_{\rm power}=4$ .

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- Vary attachment kernel.
  - Vary mechanisms
    - 1. Add edge deletio
    - 2. Add node deletion
    - 3. Add edge rewiring
- Deal with directed versus undirected networks
- Important Q.: Are there distinct universality classes for these networks?
- ▶ Qp==tow does changing the model affect
- Q.: Do-we need preferential attachment and growth
- ▶ Q. Do model details matter?
- ► The answer is (surprisingly) yes

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- Let's look at preferential attachment (PA) a little more closely.

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- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- For example: If  $P_{\text{attach}}(k) \propto k$ , we need to determine the constant of proportionality
- ➤ We need to know what everyone's degree is.
- PA is ... an outrageous assumption of node capability
- ▶ But a very simple mechanism saves the day.

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- Instead of attaching preferentially, allow new nodes to attach randomly.
- ► Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- ▶ We know that friends are weird
- ► Assuming the existing network is random, we know probability of a random friend having degree *k* is

 $Q_k \propto k P_k$ 

So rich-gets-richer scheme can now be seen to work in a natural way.

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- ► We know that friends are weird
- ▶ Assuming the existing network is random, we know probability of a random friend having degree *k* is

 $Q_k \propto kP_k$ 

So rich-gets-richer scheme can now be seen to work

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We've looked at some aspects of contagion on scale-free networks:

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- We've looked at some aspects of contagion on scale-free networks:
  - Facilitate disease-like spreading.

### Scale-Free Networks

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- We've looked at some aspects of contagion on scale-free networks:
  - 1. Facilitate disease-like spreading.
  - 2. Inhibit threshold-like spreading.
- Another simple story concerns system robustness.
- ► Albert et al. Nature 2000
  - "Error and attack tolerance of complex networks" [1]

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#### Scale-Free Networks

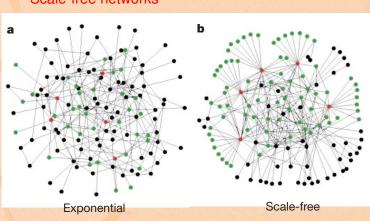
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 Standard random networks (Erdős-Rényi) versus Scale-free networks



Scale-Free Networks

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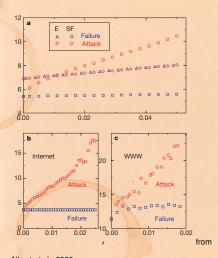
Robustness

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from



Albert et al., 2000



- Plots of network diameter as a function of fraction of nodes removed
- Erdős-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

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Albert et al., 2000

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or no
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random
- Most-connected nodes are eitherned
  - 1. Physically larger nodes that may be harder to 'target'
  - 2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes

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# Fooling with the mechanism:

2001: Redner & Krapivsky (RK) [3] explored the general attachment kernel:

**Pr**(attach to node i)  $\propto A_k = k$ 

where  $A_{\nu}$  is the attachment kernel and  $\nu > 0$ .

▶ RK also looked at changing the details of the

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▶ We'll follow RK's approach using rate equations (⊞)

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Here's the set up:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[ A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

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## Scale-Free Networks

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▶ In general, probability of attaching to a specific node of degree k at time t is

▶ E.g., for BA model,  $A_k = k$  and  $A = \sum_{k=1}^{\infty} kN_k(t)$ .

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 For  $A_k = k$ , we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time

Detail: we are ignoring initial seed network's edges.

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$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[ A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

becomes

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 $N_k = n_k t$ 

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► As for BA method, look for steady-state growing solution:

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- As for BA method, look for steady-state growing solution:  $N_k = n_k t$ .
- We replace  $dN_k/dt$  with  $dn_k t/dt = n_k$ .
- We arrive at a difference equation:

$$n_k = \frac{1}{2t}[(k-1)n_{k-1}t - kn_kt] + \delta_{k1}$$

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Rearrange and simply:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

$$k = 1 : n_1 = 2/3 \text{ since } n_0 = 0$$

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1}$$

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Rearrange and simply:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$
$$\Rightarrow (k+2)n_k = (k-1)n_{k-1} + 2\delta_{k1}$$

► Two cases:

$$k = 1 : n_1 = 2/3 \text{ since } n_0 = 0$$

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Rearrange and simply:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$
$$\Rightarrow (k+2)n_k = (k-1)n_{k-1} + 2\delta_{k1}$$

Two cases:

$$k = 1 : n_1 = 2/3 \text{ since } n_0 = 0$$

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1}$$

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Now find  $n_k$ :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1}$$

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Now find  $n_k$ :

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$

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Now find  $n_k$ :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$

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$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k+1} \frac{(k-4)}{k+1} n_{k-4}$$

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$$=\frac{(k-1)}{k+2}\frac{(k-2)}{k+1}\frac{(k-3)}{k}\frac{(k-4)}{(k-1)}\frac{(k-5)}{(k-2)}\frac{\cdots}{8}\frac{4}{7}\frac{3}{6}\frac{2}{5}\frac{1}{4}n_1$$





Now find  $n_k$ :

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
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$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} \frac{(k-4)}{(k-1)} \frac{(k-5)}{(k-2)} \cdots \frac{5}{8} \frac{4}{7} \frac{3}{6} \frac{2}{5} \frac{1}{4} n_1$$





Now find  $n_k$ :

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} \frac{(k-4)}{k-1} n_{k-4}$$

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$$= \frac{(k-1)(k-2)(k-3)(k-4)(k-5)\cdots 5}{k+2} \frac{4}{k+1} \frac{3}{k} \frac{2}{(k-1)(k-2)\cdots 8} \frac{1}{7} \frac{4}{6} \frac{3}{5} \frac{2}{4} \frac{1}{n_1}$$





Now find  $n_k$ :

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
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$$= \frac{(k-1)(k-2)(k-3)(k-4)(k-5)\cdots 54321}{k+2k+1} \frac{(k-1)(k-2)\cdots 54321}{k(k-1)(k-2)\cdots 87854} \frac{1}{8} \frac{1}{8} \frac{1}{4} \frac{1}{n_1}$$





Now find  $n_k$ :

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
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$$= \frac{(k-1)(k-2)(k-3)(k-4)(k-5)\cdots 54321}{k+2k+1} \frac{1}{k} \frac{(k-1)(k-2)\cdots 87854}{(k-1)(k-2)\cdots 87854} n_1$$

$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)}n_1$$





Now find  $n_k$ :

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
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$$= \frac{(k-1)(k-2)(k-3)(k-4)(k-5)\cdots 5\cancel{4}\cancel{3}\cancel{2}\cancel{1}}{k+2} \frac{1}{k+1} \frac{1}{k} \frac{1}{(k-1)(k-2)\cdots \cancel{8}\cancel{7}\cancel{8}\cancel{5}\cancel{4}} \frac{1}{k} \frac{1}$$

$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)}n_1 = \frac{4}{k(k+1)(k+2)}$$





Now find  $n_k$ :

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$$= \frac{(k-1)(k-2)(k-3)(k-4)(k-5)\cdots 5}{k+2} \frac{4321}{k+1} \frac{1}{k} \frac{1}{(k-1)(k-2)\cdots 87854} \frac{321}{854} n_1$$

$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)}n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$$





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As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}$$
 for large  $k$ .

- Now: what happens if we start playing around with the attachment kernel A<sub>k</sub>?
  - Again, is the result  $\gamma = 3$  universal ( $\boxplus$ )
- Natural modification:  $A_k = k^{\nu}$  with  $\nu \neq 1$ .
- But we'll first explore a more subtle modification of A<sub>n</sub> made by Redner/Krapivsky <sup>[3]</sup>
- ► Keep A<sub>k</sub> linear in k but tweak details
- ▶ Idea: Relax from  $A_k = k$  to  $A_k \sim k$  as k.

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- $\blacktriangleright$  Keep  $A_k$  linear in k but tweak details.
- ▶ Idea: Relax from  $A_k = k$  to  $A_k \sim k$  as  $k \to \infty$ .

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Recall we used the normalization:

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t$$
 for large  $t$ .

► We now have

$$A(t) = \sum_{k} A_k N_{k'}(t)$$

where we only know the asymptotic behavior of  $A_k$ 

We assume that  $A = \mu$ 

▶ We'll find µ later and make sure that our assumption is consistent.

As before, also assume  $N_k(t) = n_k t$ 

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For  $A_k = k$  we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

This now becomes

$$\Rightarrow (A_{\nu} + \mu)n_{\nu} \equiv A_{\nu} + n_{\nu} + \mu \delta_{\nu}$$

$$k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{u + A}$$

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$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

► This now becomes

$$n_k = \frac{1}{\mu} \left[ A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1}$$

$$k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

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$$n_k = \frac{1}{\mu} \left[ A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

$$\Rightarrow (\mathbf{A}_k + \mu)\mathbf{n}_k = \mathbf{A}_{k-1}\mathbf{n}_{k-1} + \mu\delta_{k1}$$

# $k = 1 : n_1 = \frac{\mu}{\mu + A_1}$

$$k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

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$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

Again two cases:

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1}.$$

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Again two cases:

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1}.$$

$$k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

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▶ Dealing with the k > 1 case:

$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

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Dealing with the k > 1 case:

$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

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Dealing with the k > 1 case:

$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_{k-2} \frac{A_{k-2}}{A_{k-1}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

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Dealing with the k > 1 case:

$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

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Dealing with the k > 1 case:

$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_{k-2} \frac{A_{k-2}}{A_{k_1}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$
$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

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▶ Dealing with the k > 1 case:

$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_{k-2} \frac{A_{k-2}}{A_{k_1}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$
$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= n_1 \frac{A_1}{A_k} \left( 1 + \frac{\mu}{A_1} \right) \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

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Dealing with the k > 1 case:

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$$= \frac{\mu}{A_k} \prod_{i=1}^k \frac{1}{1 + \frac{\mu}{A_i}}$$
 since  $n_1 = \mu/(\mu + A_1)$ 

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► Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .

$$=\frac{\mu}{A_k}\prod_{j=1}^k\frac{A_j}{A_j+\mu}$$

$$=\frac{\mu}{\cancel{A_k}}\frac{A_1}{(A_1+\mu)}\frac{A_2}{(A_2+\mu)}\cdots\frac{k-1}{(k-1+\mu)}\frac{\cancel{k}}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi}k^{k+1/2}e^{-k}}{\sqrt{2\pi}(k+\mu+1)^{k+\mu+1+1/2}e^{-(k+\mu+1)}} \\ \propto k^{-\mu-1}$$

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- ► Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
- ► For large k:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

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- Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
- ► For large k:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} = \frac{\mu}{A_k} \prod_{j=1}^k \frac{A_j}{A_j + \mu}$$

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- ► Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
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$$=\frac{\mu}{\cancel{A_k}}\frac{A_1}{(A_1+\mu)}\frac{A_2}{(A_2+\mu)}\cdots\frac{k-1}{(k-1+\mu)}\frac{\cancel{k}}{(k+\mu)}$$

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$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)}$$

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- Time for pure excitement: Find asymptotic behavior of  $n_k$  given  $A_k \to k$  as  $k \to \infty$ .
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Since μ depends on A<sub>k</sub>, details matter...

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- Now we need to find  $\mu$ .

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- Now we need to find  $\mu$ .
- Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$

$$\mu = \sum_{k=1}^{\infty} n_k A_k$$

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- Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
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$$\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

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- Now we need to find  $\mu$ .
- Our assumption again:  $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since  $N_k = n_k t$ , we have the simplification  $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for  $n_k$ :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

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$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

Closed form expression for  $\mu$ .

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- Now we need to find  $\mu$ .
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$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- Closed form expression for  $\mu$ .
- We can solve for  $\mu$  in some cases.

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- Now we need to find  $\mu$ .
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$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- Closed form expression for  $\mu$ .
- We can solve for  $\mu$  in some cases.
- ▶ Our assumption that  $A = \mu t$  is okay.

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Amazingly, we can adjust  $A_k$  and tune  $\gamma$  to be anywhere in  $[2, \infty)$ .

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- Amazingly, we can adjust  $A_k$  and tune  $\gamma$  to be anywhere in  $[2, \infty)$ .
- $\gamma = 2$  is the lower limit since

$$\mu = \sum_{k=1}^{\infty} A_k n_k \sim \sum_{k=1}^{\infty} k n_k$$

must be finite.

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- Amazingly, we can adjust  $A_k$  and tune  $\gamma$  to be anywhere in  $[2, \infty)$ .
- $\gamma = 2$  is the lower limit since

$$\mu = \sum_{k=1}^{\infty} A_k n_k \sim \sum_{k=1}^{\infty} k n_k$$

must be finite.

Let's now look at a specific example of  $A_k$  to see this range of  $\gamma$  is possible.

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▶ Consider  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .

➤ Find a = w + 1 by finding

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{i=1}^{k} \frac{1}{1 + \frac{\mu}{A_i}}$$

$$1 \frac{1}{1 + \frac{\mu}{A_1}} = \frac{1}{1 + \frac{\mu}{A_1}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$\frac{\frac{\mu}{\alpha}}{1+\frac{\mu}{\alpha}} = \frac{1}{1+\frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1+\frac{\mu}{A_j}} \text{ since } A_1 = \alpha$$

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- ▶ Consider  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- Find  $\gamma = \mu + 1$  by finding  $\mu$ .

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 \frac{1}{1 + \frac{\mu}{A_1}} = \frac{1}{1 + \frac{\mu}{A_1}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

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- ▶ Consider  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- Find  $\gamma = \mu + 1$  by finding  $\mu$ .
- $\blacktriangleright$  Expression for  $\mu$ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

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- ▶ Consider  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- Find  $\gamma = \mu + 1$  by finding  $\mu$ .
- Expression for μ:

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

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- ▶ Consider  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- Find  $\gamma = \mu + 1$  by finding  $\mu$ .
- Expression for μ:

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 - \frac{1}{1 + \frac{\mu}{A_1}} = \frac{1}{1 + \frac{\mu}{A_1}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

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- ▶ Consider  $A_1 = \alpha$  and  $A_k = k$  for  $k \ge 2$ .
- Find  $\gamma = \mu + 1$  by finding  $\mu$ .
- **Expression** for  $\mu$ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

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Carrying on:

$$\frac{\frac{\mu}{\alpha}}{1 + \frac{\mu}{\alpha}} = \frac{1}{1 + \frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

 $\frac{\mu}{\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$ 

Now her rocult that [3]

$$\frac{F(a+k)}{F(b+k)} = \frac{F(a+2)}{(b-a-1)F(b+1)}$$

 $\frac{\Gamma(3)}{1-1-1)\Gamma(2+\mu)}\Gamma(2+\mu)$ 

$$\Rightarrow \mu(\mu - 1) = 2\alpha$$

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Carrying on:

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$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

► Now use result that [3]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

with a = 1 and  $b = \mu + 1$ .

## $\Rightarrow \mu(\mu - 1) = 2\alpha$

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Carrying on:

$$\frac{\frac{\mu}{\alpha}}{\frac{1}{1+\frac{\mu}{\alpha}}} = \frac{1}{\frac{1}{1+\frac{\mu}{\alpha}}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1+\frac{\mu}{A_j}}$$

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

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with a = 1 and  $b = \mu + 1$ .

$$\mu = \alpha \frac{\Gamma(3)}{(\mu + 1 - 1 - 1)\Gamma(2 + \mu)} \Gamma(2 + \mu)$$

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Carrying on:

$$\frac{\frac{\mu}{\alpha}}{1+\frac{\mu}{\alpha}} = \frac{1}{1+\frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1+\frac{\mu}{A_j}}$$

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$$\Rightarrow \mu(\mu - 1) = 2\alpha$$

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# Universality?

$$\mu(\mu-1)=2\alpha\Rightarrow\mu=\frac{1+\sqrt{1+8\alpha}}{2}$$

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# Universality?

$$\mu(\mu-1)=2\alpha\Rightarrow\mu=\frac{1+\sqrt{1+8\alpha}}{2}.$$

▶ Since  $\gamma = \mu + 1$ , we have

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

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# Universality?

$$\mu(\mu-1)=2\alpha\Rightarrow\mu=\frac{1+\sqrt{1+8\alpha}}{2}.$$

Since  $\gamma = \mu + 1$ , we have

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

Craziness...

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Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with  $0 < \nu < 1$ .

General finding by Krapivsky and Redner: [9

- Stretched exponentials (truncated power laws)
- aka Weibull distribution
- Universality yow details of kernel do not matter
- ➤ Distribution of degree is universal providing a

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### Details:

▶ For  $1/2 < \nu < 1$ :

$$n_k \sim k^{-\nu} e^{-\mu \left(rac{k^{1-
u}-2^{1-
u}}{1-
u}
ight)}$$

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### Details:

▶ For  $1/2 < \nu < 1$ :

$$n_k \sim k^{-\nu} e^{-\mu \left(\frac{k^{1-\nu}-2^{1-\nu}}{1-\nu}\right)}$$

For  $1/3 < \nu < 1/2$ :

$$n_k \sim k^{-\nu} e^{-\mu rac{k^{1-
u}}{1-
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And for  $1/(r+1) < \nu < 1/r$ , we have r pieces in exponential.

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### Outline

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Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with  $\nu > 1$ .

- Now a winner-take-all mechanism
- One single node ends up being connected to almost all other nodes.
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