Generating Functions for Random Networks

Complex Networks CSYS/MATH 303, Spring, 2011

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 Idea: Given a sequence a₀, a₁, a₂,..., associate each element with a distinct function or other mathematical object.

sequences and retrieve sequence elements. Definition:

► The generating function (g.f.) for a sequence {a_n} is

$$F(x)=\sum_{n=0}^{\infty}a_nx^n.$$

 Roughly: transforms a vector in R[∞] into a functior defined on R¹.

Related to Fourier, Laplace, Mellin, ...

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- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

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Simple example

Rolling dice:

▶ $p_k^{(\square)} = \mathbf{Pr}(\text{throwing a } k) = 1/6 \text{ where } k = 1, 2, ..., 6.$

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We'll come back to this simple example as we derive various dehotous properties of generating functions.



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$$F^{(\Box)}(x) = \sum_{k=1}^{6} p_k x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

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Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where $c = 1 - e^{-\lambda}$.

$$x^{k} = \sum_{k=0}^{\infty} c e^{-\lambda k} x^{k} = rac{c}{1 - x e^{-\lambda k}}$$





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$$=\sum_{k=0}^{\infty}P_k=1$$

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• Notice that $F(1) = c/(1 - e^{-\lambda}) = 1$.

$k = \sum_{k=0}^{\infty} P_k = 1$

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 For probability distributions, we must always have F(1) = 1 since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k$$

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Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k$$

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Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \Big|_{x=1}$$

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 In general, many calculations become simple, if a little abstract.



Average degree:

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$$= \frac{d}{dx} F(x) \bigg|_{x=1} = F(1)$$

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- In general, many calculations become simple, if a little abstract.
- For our exponential example:

$$F'(x)=\frac{(1-e^{-\lambda})e^{-\lambda}}{(1-xe^{-\lambda})^2}.$$



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Average degree:

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Properties of generating functions Useful pieces for probability distributions:

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Properties of generating functions Useful pieces for probability distributions:

F(1) = 1

Normalization:



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Properties of generating functions Useful pieces for probability distributions:

Normalization:

First moment:

 $\langle k \rangle = F'(1)$

F(1) = 1

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Useful pieces for probability distributions:

Normalization:

First moment:

$$\langle k \rangle = F'(1)$$

F(1) = 1

Higher moments:

$$\langle k^n \rangle = \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^n F(x) \Big|_{x=1}$$

$$\mathsf{P}_k = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) \Big|_{\mathcal{S}}$$

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Useful pieces for probability distributions:

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F(1) = 1

Higher moments:

 $\langle k^n \rangle = \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^n F(x) \Big|_{x=1}$

kth element of sequence (general):

$$P_k = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) \Big|_{x=1}$$

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Edge-degree distribution

Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

Let's reexpress our condition in terms of ge functions.
We first need the g.f. for *R_k*.
We'll now use this notation: *F_P(x)* is the g.f. for *P_k*. *F_R(x)* is the g.f. for *R_k*.
Condition in terms of g.f. is:

Now find how F_R is related to F_P.

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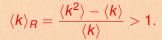
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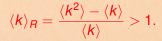
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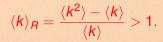
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Condition in terms of g.f. is:

 $\langle k \rangle_R = F'_R(1) > 1.$

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Now find how F_R is related to F_P...

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We have

$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k x^k}{k!}$$

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We have

$$F_R(x) = \sum_{k=0}^{\infty} \frac{\mathbf{R}_k}{\mathbf{R}_k} x^k = \sum_{k=0}^{\infty} \frac{(k+1)\mathbf{P}_{k+1}}{\langle k \rangle} x^k.$$

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Shift index to j = k + 1 and pull out $\frac{1}{\langle k \rangle}$:

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$$F_R(x) = \sum_{k=0}^{\infty} \frac{R_k x^k}{k} = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to j = k + 1 and pull out $\frac{1}{\langle k \rangle}$:

$$F_R(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1}$$

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$$= \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{\mathrm{d}}{\mathrm{d}x} \left(F_P(x) - P_0 \right)$$

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Finally, since $\langle k \rangle = F'_P(1)$,

$$F_R(x) = \frac{F'_P(x)}{F'_P(1)}$$

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• Recall giant component condition is $\langle k \rangle_R = F'_R(1) > 1.$

 $F'_{R}(x) = \frac{F''_{P}(x)}{F'_{P}(1)}$

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- Since we have $F_R(x) = F'_P(x)/F'_P(1)$,

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Since we have $F_R(x) = F'_P(x)/F'_P(1)$,

$$F'_{R}(x) = rac{F''_{P}(x)}{F'_{P}(1).}$$

Setting x = 1, our condition becomes

$$\frac{F_P''(1)}{F_P'(1)} > 1$$

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To figure out the size of the largest component (S_1) , we need more resolution on component sizes.

Definitions:

- π_n = probability that a random node belongs to a finite component of size $n < \infty$.
- ρ_n = probability a random link leads to a finite subcomponent of size n < ∞.

Local-global connection:

 $r_k, n_k \Leftrightarrow \pi_n, \rho_n$ neighbors \Leftrightarrow components

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The largest component:

Subtle key: F_π(1) is the probability that a node belongs to a finite component.
 Therefore: S₁ = 1 - F_π(1).

Our mission, which we accept:

Find the four generating functions

 F_P, F_R, F_π , and F_ρ

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Sneaky Result 1:

Consider two random variables U and V whose values may be 0, 1, 2,
 Write probability distributions as U_k and V_k and (as F_U and F_V)
 SR1. If a third random variable is defined as

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then

 $F_W(x) = F_U(F_V(x))$



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 $W = \sum_{i=1}^{D} V^{(i)}$ with each $V^{(i)} \stackrel{d}{=} V$

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Proof of SR1:

Write probability that variable W has value k as W_k .

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Proof of SR1:

Write probability that variable W has value k as W_k .

$$W_k = \sum_{j=0}^{\infty} U_j \times \Pr(\text{sum of } j \text{ draws of variable } V = k)$$

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Write probability that variable W has value k as W_k .

$$W_k = \sum_{j=0}^{\infty} U_j imes$$
 Pr(sum of *j* draws of variable $V = k_j^2$

$$=\sum_{j=0}^{\infty} U_{j} \sum_{\substack{\{i_{1},i_{2},\ldots,i_{j}\} \mid \\ i_{1}+i_{2}+\ldots+i_{j}=k}} V_{i_{1}} V_{i_{2}} \cdots V_{i_{j}}$$

$$\therefore F_{W}(x) = \sum_{k=0}^{\infty} W_{k} x^{k} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} U_{j} \sum_{\substack{\{i_{1}, i_{2}, \dots, i_{j}\} \mid \\ i_{1}+i_{2}+\dots+i_{j}=k}} V_{i_{1}} V_{i_{2}} \cdots V_{i_{j}} x^{k}$$

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 $=\sum_{j=0}U_j\sum_{k=0}$

Write probability that variable W has value k as W_k .

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$$=\sum_{j=0}^{\infty} U_{j} \sum_{k=0}^{\infty} \sum_{\substack{\{i_{1},i_{2},...,i_{j}\} \mid \\ i_{1}+i_{2}+...+i_{j}=k}} V_{i_{1}} x^{i_{1}} V_{i_{2}} x^{i_{2}} \cdots V_{i_{j}} x^{i_{j}}$$

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With some concentration, observe:

$$F_{W}(x) = \sum_{j=0}^{\infty} U_{j} \sum_{k=0}^{\infty} \underbrace{\sum_{\substack{\{i_{1},i_{2},...,i_{j}\} \mid \\ i_{1}+i_{2}+...+i_{j}=k}} V_{i_{1}} x^{i_{1}} V_{i_{2}} x^{i_{2}} \cdots V_{i_{j}} x^{i_{j}}}_{x^{k} \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^{j}}$$

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With some concentration, observe:

$$F_{W}(x) = \sum_{j=0}^{\infty} U_{j} \sum_{k=0}^{\infty} \sum_{\substack{\{i_{1},i_{2},...,i_{j}\} \mid \\ i_{1}+i_{2}+...+i_{j}=k}} V_{i_{1}} x^{i_{1}} V_{i_{2}} x^{i_{2}} \cdots V_{i_{j}} x^{i_{j}}} x^{i_{j}}}{x^{k} \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'}\right)^{j}} \left(\sum_{j'=0}^{\infty} V_{j'} x^{i'}\right)^{j}} = (F_{V}(x))^{j}}$$
$$= \sum_{j=0}^{\infty} U_{j} (F_{V}(x))^{j}$$
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SR2 If a second random variable is defined as $V = U + 1 \text{ then } F_V(x) = xF_U(x)$

• Reason: $V_k = U_{k-1}$ for $k \ge 1$ and $V_0 = 0$

 $= x \sum_{j=0} U_j x^j = x F_U(x) \cdot v$

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Start with a random variable U with distribution U_k (k = 0, 1, 2, ...)

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 $x^{k} = \sum_{k=1}^{k} U_{k-1} x$

 $= x \sum_{j=0} U_j x^j = x F_U(x) . \mathbf{v}$

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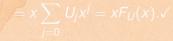
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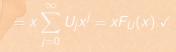
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Generalization of SR2:

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 $= x^{-i} \sum_{k=0}^{\infty} U_k x^k$



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Generalization of SR2:

(1) If V = U + i then

 $F_V(x) = x^i F_U(x).$

$F_V(x) = x^{-i} F_U(x)$



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Generalization of SR2:

(1) If V = U + i then

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$$=x^{-i}\sum_{k=0}^{\infty}U_kx^k$$

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• Goal: figure out forms of the component generating functions, F_{π} and F_{ρ} .

 $\int P_k \times \Pr\left(\begin{array}{c} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array}\right)$



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- Goal: figure out forms of the component generating functions, F_{π} and F_{ρ} .
- π_n = probability that a random node belongs to a finite component of size *n*

um of sizes of subcomponents t end of *k* random links = *n* -



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Therefore:





- Goal: figure out forms of the component generating functions, F_{π} and F_{ρ} .
- π_n = probability that a random node belongs to a finite component of size *n*

 $= \sum_{k=0}^{\infty} P_k \times \Pr\left(\begin{array}{c} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array}\right)$

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Therefore:

$$F_{\pi}(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_{P}(F_{\rho}(x))}_{\text{SR1}}$$



- Goal: figure out forms of the component generating functions, F_{π} and F_{ρ} .
- π_n = probability that a random node belongs to a finite component of size *n*

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Therefore:
$$F_{\pi}(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_{P}(F_{\rho}(x))}_{\text{SR1}}$$

Extra factor of x accounts for random node itself.

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ρ_n = probability that a random link leads to a finite subcomponent of size n.

following a random edge, the outgoing edges of node reached lead to finite subcomponents of combined size *n* = 1.

 \times Pr $\begin{pmatrix} \text{sum of sizes of subcomponent} \\ \text{at end of } k \text{ random links} = n - \end{pmatrix}$



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- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- Invoke one step of recursion: ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size n 1,

im of sizes of subcomponents end of k random links = n - 1

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- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- lnvoke one step of recursion: ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size n-1.

 $= \sum_{k=0}^{\infty} R_k \times \Pr\left(\begin{array}{c} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array}\right)$

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- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- Invoke one step of recursion: ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size n 1,

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Therefore:



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- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- Invoke one step of recursion: ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size n 1,

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The

refore:
$$F_{\rho}(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_{R}(F_{\rho}(x))}_{\text{SR1}}$$

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- ρ_n = probability that a random link leads to a finite subcomponent of size n.
- Invoke one step of recursion: ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size n 1,

 $= \sum_{k=0}^{\infty} R_k \times \Pr\left(\begin{array}{c} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n-1 \end{array} \right)$

Therefore:
$$F_{\rho}(x) = \underbrace{x}_{SR2} \underbrace{F_{R}(F_{\rho}(x))}_{SR1}$$

Again, extra factor of x accounts for random node itself.

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We now have two functional equations connecting our generating functions:

 $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$ and $F_{\rho}(x) = xF_{R}(F_{\rho}(x))$

We first untangle the second equation to find F_p
 We can do this because it only involves F_p and F
 The first equation then immediately gives us F_π in terms of F_p and F_p.

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We now have two functional equations connecting our generating functions:

 $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$ and $F_{\rho}(x) = xF_{R}(F_{\rho}(x))$

► Taking stock: We know $F_P(x)$ and $F_R(x) = F'_P(x)/F'_P(1)$.

second equation to fir

Ve can do this because it only involves F_p and F the first equation then immediately gives us F_π in

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We first untangle the second equation to find F_ρ

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- We first untangle the second equation to find F_ρ
- We can do this because it only involves F_{ρ} and F_{R} .

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We now have two functional equations connecting our generating functions:

 $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$ and $F_{\rho}(x) = xF_{R}(F_{\rho}(x))$

- ► Taking stock: We know $F_P(x)$ and $F_R(x) = F'_P(x)/F'_P(1)$.
- We first untangle the second equation to find F_ρ
- We can do this because it only involves F_{ρ} and F_{R} .
- The first equation then immediately gives us F_{π} in terms of F_{ρ} and F_{R} .

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Remembering vaguely what we are doing:

Finding F_x to obtain the fractional size of the larges component $S_1 = 1 - F_x(1)$. Set x = 1 in our two equations:

 $F_{\pi}(1) = F_{P}(F_{\rho}(1))$ and $F_{\rho}(1) = F_{R}(F_{\rho}(1))$

Solve second equation numerically for $F_{\rho}(1 \ge \text{Plug } F_{\rho}(1))$ into first equation to obtain $F_{\gamma}(1)$

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Remembering vaguely what we are doing:
 Finding *F*_π to obtain the fractional size of the largest component *S*₁ = 1 - *F*_π(1).

 $F_{\pi}(1) = F_{P}(F_{\rho}(1))$ and $F_{\rho}(1) = F_{R}(F_{\rho}(1))$

Solve second equation numerically for $F_{\rho}(1)$ Plug $F_{\rho}(1)$ into first equation to obtain $F_{\tau}(1)$

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Remembering vaguely what we are doing:
 Finding *F_π* to obtain the fractional size of the largest component *S*₁ = 1 - *F_π*(1).

Set x = 1 in our two equations:

 $F_{\pi}(1) = F_{P}(F_{\rho}(1))$ and $F_{\rho}(1) = F_{R}(F_{\rho}(1))$

Solve second equation numerically for F_a(1
 Plug F_a(4) into first equation to obtain F_a(1

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Remembering vaguely what we are doing:

Finding F_{π} to obtain the fractional size of the largest component $S_1 = 1 - F_{\pi}(1)$.

Set x = 1 in our two equations:

 $F_{\pi}(1) = F_{P}(F_{\rho}(1))$ and $F_{\rho}(1) = F_{R}(F_{\rho}(1))$

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Remembering vaguely what we are doing:

Finding F_{π} to obtain the fractional size of the largest component $S_1 = 1 - F_{\pi}(1)$.

Set x = 1 in our two equations:

 $F_{\pi}(1) = F_{P}(F_{\rho}(1))$ and $F_{\rho}(1) = F_{R}(F_{\rho}(1))$

Solve second equation numerically for $F_{\rho}(1)$.

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Remembering vaguely what we are doing:

Finding F_{π} to obtain the fractional size of the largest component $S_1 = 1 - F_{\pi}(1)$.

Set x = 1 in our two equations:

 $F_{\pi}(1) = F_{P}(F_{\rho}(1))$ and $F_{\rho}(1) = F_{R}(F_{\rho}(1))$

Solve second equation numerically for F_ρ(1).
 Plug F_ρ(1) into first equation to obtain F_π(1).

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Example: Standard random graphs.

• We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

 $F_P(x)$...aha



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Example: Standard random graphs. We can show F_P(x) = e^{-⟨k⟩(1-x)}

 $\therefore F_R(x) = F'_P(x)/F'_P(1)$

$=F_P(x)$...aha



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Example: Standard random graphs.

• We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

:
$$F_R(x) = F'_P(x)/F'_P(1) = e^{-\langle k \rangle (1-x)}/e^{-\langle k \rangle (1-x')}|_{x'=x}$$

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$=F_P(x)$...aha!

► FIHS's erour two equations are the same.
 So F_π(x) = F_µ(x) = xF_B(F_µ(x)) = xF_B(F_π(x))
 ► Why our dirty (but wrong) trick worked earlied



Example: Standard random graphs.

• We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\therefore F_R(x) = F'_P(x)/F'_P(1) = e^{-\langle k \rangle (1-x)}/e^{-\langle k \rangle (1-x')}|_{x'=1}$$

 $= e^{-\langle k \rangle (1-x)} = F_P(x)$...aha!



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Example: Standard random graphs.

• We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\therefore F_R(x) = F'_P(x)/F'_P(1) = e^{-\langle k \rangle (1-x)}/e^{-\langle k \rangle (1-x')}|_{x'=1}$$

 $= e^{-\langle k \rangle (1-x)} = F_P(x)$...aha!



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Example: Standard random graphs.

• We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\therefore F_R(x) = F'_P(x)/F'_P(1) = e^{-\langle k \rangle (1-x)}/e^{-\langle k \rangle (1-x')}|_{x'=1}$$

$$=e^{-\langle k \rangle(1-x)}=F_P(x)$$
 ...aha!

RHS's of our two equations are the same.

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Example: Standard random graphs.

• We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\therefore F_R(x) = F'_P(x)/F'_P(1) = e^{-\langle k \rangle (1-x)}/e^{-\langle k \rangle (1-x')}|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)} = F_P(x)$$
 ...aha!

PHS's of our two equations are the same. So F_π(x) = F_ρ(x) = xF_R(F_ρ(x)) = xF_R(F_π(x))

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Example: Standard random graphs.

• We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\therefore F_R(x) = F'_P(x)/F'_P(1) = e^{-\langle k \rangle (1-x)}/e^{-\langle k \rangle (1-x')}|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)} = F_P(x)$$
 ...aha!

RHS's of our two equations are the same.
 So F_π(x) = F_ρ(x) = xF_R(F_ρ(x)) = xF_R(F_π(x))
 Why our dirty (but wrong) trick worked earlier...

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• We are down to $F_{\pi}(x) = xF_{B}(F_{\pi}(x))$ and $F_{B}(x) = e^{-\langle k \rangle (1-x)}$.

 $1-S_1=e^{-\langle k\rangle S_1}$

Or: $\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$

> Just as we found with our dirty trick

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• We are down to $F_{\pi}(x) = xF_{R}(F_{\pi}(x))$ and $F_{R}(x) = e^{-\langle k \rangle (1-x)}$.

$$\therefore$$
 $F_{\pi}(x) = xe^{-\langle k \rangle (1 - F_{\pi}(x))}$



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Just as we found with our dirty trick



• We are down to $F_{\pi}(x) = xF_{R}(F_{\pi}(x))$ and $F_{R}(x) = e^{-\langle k \rangle (1-x)}$. $\therefore F_{\pi}(x) = xe^{-\langle k \rangle (1-F_{\pi}(x))}$ • We're first after $S_{1} = 1 - F_{\pi}(1)$ so set x = 1 and replace $F_{\pi}(1)$ by $1 - S_{1}$:



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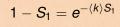
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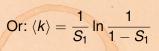
References

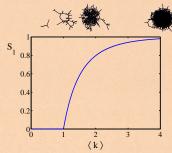


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• We are down to $F_{\pi}(x) = xF_{R}(F_{\pi}(x))$ and $F_{R}(x) = e^{-\langle k \rangle (1-x)}$. $\therefore F_{\pi}(x) = xe^{-\langle k \rangle (1-F_{\pi}(x))}$ • We're first after $S_{1} = 1 - F_{\pi}(1)$ so set x = 1 and replace $F_{\pi}(1)$ by $1 - S_{1}$:







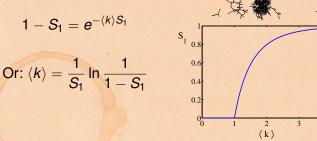
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• We are down to $F_{\pi}(x) = xF_{R}(F_{\pi}(x))$ and $F_{R}(x) = e^{-\langle k \rangle (1-x)}$. $\therefore F_{\pi}(x) = xe^{-\langle k \rangle (1-F_{\pi}(x))}$ • We're first after $S_{1} = 1 - F_{\pi}(1)$ so set x = 1 and replace $F_{\pi}(1)$ by $1 - S_{1}$:



Just as we found with our dirty trick ...

Generating Functions

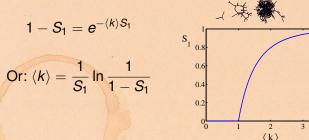
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• We are down to $F_{\pi}(x) = xF_{R}(F_{\pi}(x))$ and $F_{R}(x) = e^{-\langle k \rangle (1-x)}$. $\therefore F_{\pi}(x) = xe^{-\langle k \rangle (1-F_{\pi}(x))}$ • We're first after $S_{1} = 1 - F_{\pi}(1)$ so set x = 1 and replace $F_{\pi}(1)$ by $1 - S_{1}$:



- Just as we found with our dirty trick ...
- Again, we (usually) have to resort to numerics ...

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• Next: find average size of finite components $\langle n \rangle$.

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- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.

Starting from $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$, we differentia $F'_{\pi}(x) = F_{P}(F_{\rho}(x)) + xF'_{\rho}(x)F'_{P}(F_{\rho}(x))$

• While $F_{\rho}(x) = xF_{R}(F_{\rho}(x))$ gives

 $= F_R(F_\rho(x)) + xF'_\rho(x)F'_R(F_\rho(x))$

Now set x > 1 in both equations.
 We solve the second equation for F'₀(1) (we musilized, have F₀(1)).

Plug F((1) and F((1) into first equation to find F

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- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.
- Try to avoid finding $F_{\pi}(x)$...

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- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.
- Try to avoid finding $F_{\pi}(x)$...

Starting from $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$, we differentiate:

 $F'_{\pi}(x) = F_{\mathcal{P}}\left(F_{\rho}(x)\right) + xF'_{\rho}(x)F'_{\mathcal{P}}\left(F_{\rho}(x)\right)$

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- Next: find average size of finite components $\langle n \rangle$.
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 $F'_{\pi}(x) = F_{\mathcal{P}}\left(F_{\rho}(x)\right) + xF'_{\rho}(x)F'_{\mathcal{P}}\left(F_{\rho}(x)\right)$

• While $F_{\rho}(x) = xF_R(F_{\rho}(x))$ gives

 $F'_{\rho}(x) = F_{R}\left(F_{\rho}(x)\right) + xF'_{\rho}(x)F'_{R}\left(F_{\rho}(x)\right)$

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Average Component Size



- Next: find average size of finite components $\langle n \rangle$.
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 $F'_{\rho}(x) = F_R(F_{\rho}(x)) + xF'_{\rho}(x)F'_R(F_{\rho}(x))$

Now set x = 1 in both equations.

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Average Component Size



- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.
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• While $F_{\rho}(x) = xF_R(F_{\rho}(x))$ gives

 $F'_{\rho}(x) = F_R(F_{\rho}(x)) + xF'_{\rho}(x)F'_R(F_{\rho}(x))$

- Now set x = 1 in both equations.
- We solve the second equation for F'_ρ(1) (we must already have F_ρ(1)).

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- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_{\pi}(1)$.
- Try to avoid finding $F_{\pi}(x)$...

Starting from $F_{\pi}(x) = xF_{P}(F_{\rho}(x))$, we differentiate:

 $F'_{\pi}(x) = F_{\mathcal{P}}\left(F_{\rho}(x)\right) + xF'_{\rho}(x)F'_{\mathcal{P}}\left(F_{\rho}(x)\right)$

• While $F_{\rho}(x) = xF_R(F_{\rho}(x))$ gives

 $F'_{\rho}(x) = F_{R}(F_{\rho}(x)) + xF'_{\rho}(x)F'_{R}(F_{\rho}(x))$

- Now set x = 1 in both equations.
- We solve the second equation for F'_ρ(1) (we must already have F_ρ(1)).

• Plug $F'_{\rho}(1)$ and $F_{\rho}(1)$ into first equation to find $F'_{\pi}(1)$.

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Average component size Example: Standard random graphs.

Rearrange: $F'_{\pi}(x)=rac{F_{P}\left(F_{\pi}(x) ight)}{1-xF'_{P}\left(F_{\pi}(x) ight)}$

place $F_{\pi}(x)$ using $F_{\pi}(x) = xF_{P}(F_{\pi}(x))$ by $F_{\pi}(x)$ using $F_{\pi}(x) = xF_{P}(F_{\pi}(x))$. by = 1 and replace $F_{\pi}(1)$ with $1 = S_{1}$. End result: $\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_{1})}{1 - \langle k \rangle (1 - S_{1})}$

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Average component size Example: Standard random graphs.

• Use fact that $F_P = F_R$ and $F_{\pi} = F_{\rho}$.

Rearrange: $F'_{\pi}(x) = rac{F_{\mathcal{P}}\left(F_{\pi}(x)
ight)}{1 - xF'_{\mathcal{P}}\left(F_{\pi}(x)
ight)}$

blace $F_P(F_{\pi}(x))$ using $F_{\pi}(x) = xF_P(F_{\pi}(x))$ x = 1 and replace $F_{\pi}(1)$ with $1 - S_1$. End result: $\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$

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Example: Standard random graphs.

• Use fact that $F_P = F_R$ and $F_{\pi} = F_{\rho}$.

Two differentiated equations reduce to only one:

 $F'_{\pi}(x) = F_{P}\left(F_{\pi}(x)\right) + xF'_{\pi}(x)F'_{P}\left(F_{\pi}(x)\right)$

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End result: $\langle n \rangle = F'_{\pi}(1) = rac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$



Example: Standard random graphs.

• Use fact that $F_P = F_R$ and $F_{\pi} = F_{\rho}$.

Two differentiated equations reduce to only one:

 $F'_{\pi}(x) = F_{\mathcal{P}}\left(F_{\pi}(x)\right) + xF'_{\pi}(x)F'_{\mathcal{P}}\left(F_{\pi}(x)\right)$

Rearrange:
$$F'_{\pi}(x) = rac{F_P(F_{\pi}(x))}{1 - xF'_P(F_{\pi}(x))}$$

ace $F_{P}(F_{\pm}(x))$ using $F_{\pi}(x) = xF_{P}(F_{\pm}(x))$. Let i and replace $F_{\pm}(1)$ with $1 - S_{1}$. End result: $\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_{1})}{1 - \langle k \rangle (1 - S_{1})}$

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Example: Standard random graphs.

• Use fact that $F_P = F_R$ and $F_{\pi} = F_{\rho}$.

Two differentiated equations reduce to only one:

$$F'_{\pi}(x) = F_{P}(F_{\pi}(x)) + xF'_{\pi}(x)F'_{P}(F_{\pi}(x))$$

Rearrange:
$$F'_{\pi}(x) = rac{F_P(F_{\pi}(x))}{1 - xF'_P(F_{\pi}(x))}$$

Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$

End result: $\langle n \rangle = F'_{\pi}(1) = \frac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$

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Example: Standard random graphs.

• Use fact that $F_P = F_R$ and $F_{\pi} = F_{\rho}$.

Two differentiated equations reduce to only one:

$$F'_{\pi}(x) = F_{\mathcal{P}}\left(F_{\pi}(x)\right) + xF'_{\pi}(x)F'_{\mathcal{P}}\left(F_{\pi}(x)\right)$$

Rearrange:
$$F'_{\pi}(x) = rac{F_P(F_{\pi}(x))}{1 - xF'_P(F_{\pi}(x))}$$

Simplify denominator using F'_P(x) = ⟨k⟩F_P(x)
 Replace F_P(F_π(x)) using F_π(x) = xF_P(F_π(x)).

End result: $\langle n \rangle = F'_{\pi}(1) = \frac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$

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Example: Standard random graphs.

• Use fact that $F_P = F_R$ and $F_{\pi} = F_{\rho}$.

Two differentiated equations reduce to only one:

$$F'_{\pi}(x) = F_{\mathcal{P}}\left(F_{\pi}(x)\right) + xF'_{\pi}(x)F'_{\mathcal{P}}\left(F_{\pi}(x)\right)$$

Rearrange:
$$F'_{\pi}(x) = rac{F_P(F_{\pi}(x))}{1 - xF'_P(F_{\pi}(x))}$$

- Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$
- Replace $F_P(F_{\pi}(x))$ using $F_{\pi}(x) = xF_P(F_{\pi}(x))$.
- Set x = 1 and replace $F_{\pi}(1)$ with $1 S_1$.

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Our result for standard random networks:

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References

Recall that $\langle k \rangle = 1$ is the critical value of averag degree for standard random networks. Look at what happens when we increase $\langle k \rangle$ to from below.

• We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

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This blows up as (
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component sizes at (/

Typical critical point behavio



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$$\langle n \rangle = F'_{\pi}(1) = \frac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$$

Atra on largest component size
For ⟨k⟩ = 1, S₁ ~ N^{2/3}.
For ⟨k⟩ < 1, S₁ ~ log N.

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$$\langle n \rangle = F'_{\pi}(1) = \frac{(1-S_1)}{1-\langle k \rangle(1-S_1)}$$

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• As
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