Mixed, correlated random networks

Complex Networks
CSYS/MATH 303, Spring, 2011

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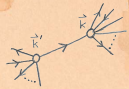
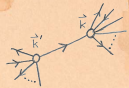
Vermont Advanced Computing Center
University of Vermont



Outline



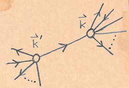
Random directed networks:

- So far, we've studied networks with undirected, unweighted edges.
- Now consider directed, unweighted edges.
- Nodes have k_i and k_o incoming and outgoing edges, otherwise random.
- Network defined by joint in- and out-degree distribution: P_{k_i,k_o}
- Normalization: $\sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} P_{k_i,k_o} = 1$
- Marginal in-degree and out-degree distributions:

$$P_{k_i} = \sum_{k_o=0}^{\infty} P_{k_i,k_o} \text{ and } P_{k_o} = \sum_{k_i=0}^{\infty} P_{k_i,k_o}$$

- Required balance:

$$\langle k_i \rangle = \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} k_i P_{k_i,k_o} = \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} k_o P_{k_i,k_o} = \langle k_o \rangle$$



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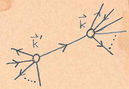

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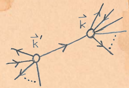

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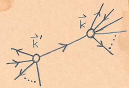

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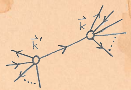

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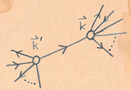

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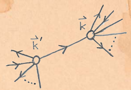

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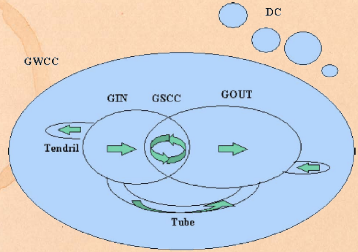

From Boguñá and Serano. [1]

- GWCC = Giant Weakly Connected Component (directions removed);
- GIN = Giant In-Component;
- GOUT = Giant Out-Component;
- GSCC = Giant Strongly Connected Component;
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- When moving through a family of increasingly connected directed random networks, GWCC usually appears before GIN, GOUT, and GSCC which tend to appear together. [4, 1]

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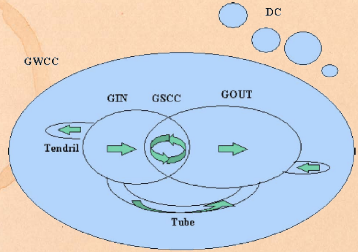

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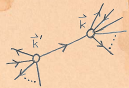

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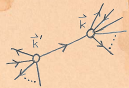
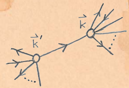


Mixed random networks:

Important observation:

- Directed and undirected random networks are separate families...
- ... and analyses are also disjoint.
- Need to examine a larger family of random networks with mixed directed and undirected edges.



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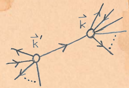

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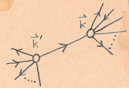

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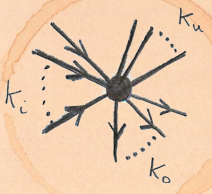
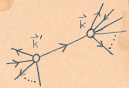


- Consider nodes with three types of edges:
  1. $k_{\rm u}$ undirected edges,
  2. $k_{\rm i}$ incoming directed edges,
  3. $k_{\rm o}$ outgoing directed edges.
- Define a node by generalized degree:

$$\vec{k} = [\, k_{\rm u} \;\; k_{\rm i} \;\; k_{\rm o} \,]^{\rm T}.$$



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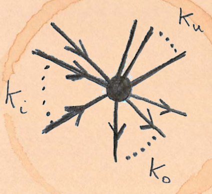

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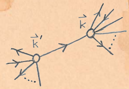

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- Joint degree distribution:

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- Otherwise, no other restrictions and connections are random.
- Directed and undirected random networks are disjoint subfamilies:

$$\text{Undirected: } P_{\vec{k}} = P_{k_{\mathrm{u}}} \delta_{k_{\mathrm{i}},0} \delta_{k_{\mathrm{o}},0},$$

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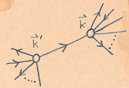

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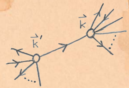

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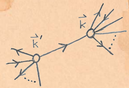

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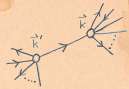

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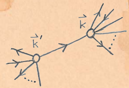

Correlations:

- Now add correlations (two point or Markovian):
  1. $P^{(u)}(\vec{k} \mid \vec{k}')$ = probability that an undirected edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node.
  2. $P^{(i)}(\vec{k} \mid \vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an in-directed edge relative to the destination node.
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- Now require more refined (detailed) balance.
- Conditional probabilities cannot be arbitrary.
  1. $P^{(u)}(\vec{k} \mid \vec{k}')$ must be related to $P^{(u)}(\vec{k}' \mid \vec{k})$.
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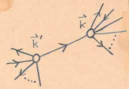

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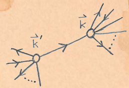

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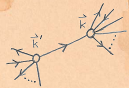

Correlations:

- Now add correlations (two point or Markovian):
  1. $P^{(\mathrm{u})}(\vec{k} \mid \vec{k}')$ = probability that an undirected edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node.
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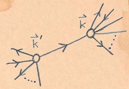

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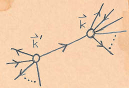

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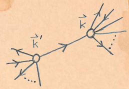

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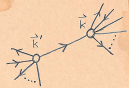

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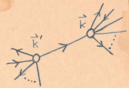

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Correlations—Undirected edge balance:

- Randomly choose an edge, and randomly choose one end.
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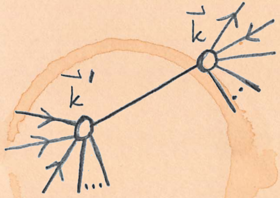

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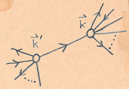

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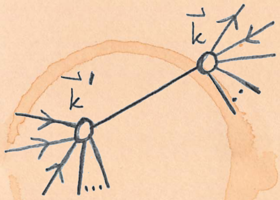

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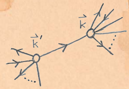

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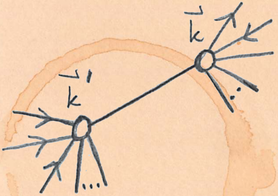

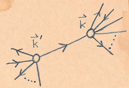

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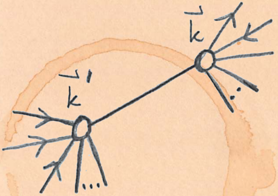

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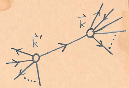

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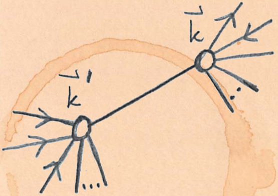

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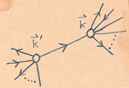


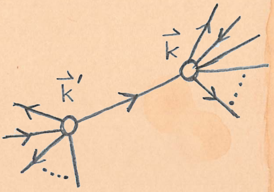
Correlations—Directed edge balance:

- The quantities

$$\frac{k_{\rm o}P(\vec{k})}{\langle k_{\rm o}\rangle} \quad \text{and} \quad \frac{k_{\rm i}P(\vec{k})}{\langle k_{\rm i}\rangle}$$

give the probabilities that in starting at a random end of a randomly selected edge, we begin at a degree $\vec{k}$ node and then find ourselves travelling:

1. along an outgoing edge, or
2. against the direction of an incoming edge.



- We therefore have

$$P^{(\rm dir)}(\vec{k},\vec{k}') = P^{(\rm i)}(\vec{k}\,|\,\vec{k}')\frac{k_{\rm o}'P(\vec{k}')}{\langle k_{\rm o}'\rangle} = P^{(\rm o)}(\vec{k}'\,|\,\vec{k})\frac{k_{\rm i}P(\vec{k})}{\langle k_{\rm i}\rangle}.$$

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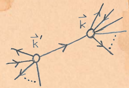

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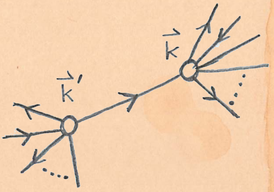

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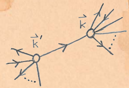

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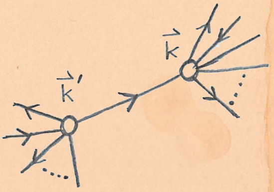

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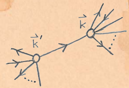
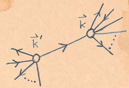


Outline



Global spreading condition: [2]

When are cascades possible?:

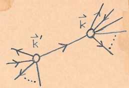
- Consider uncorrelated mixed networks first.
- Recall our first result for undirected random networks, that edge gain ratio must exceed 1:

$$\mathbf{R} = \sum_{k_u=0}^{\infty} \frac{k_u P_{k_u}}{\langle k_u \rangle} \bullet (k_u - 1) \bullet B_{k_u,1} > 1.$$

- Similar form for purely directed networks:

$$\mathbf{R} = \sum_{k_i=0}^{\infty} \sum_{k_o=0}^{\infty} \frac{k_i P_{k_i,k_o}}{\langle k_i \rangle} \bullet k_o \bullet B_{k_i,1} > 1.$$

- Both are composed of (1) probability of connection to a node of a given type; (2) number of newly infected edges if successful; and (3) probability of infection.



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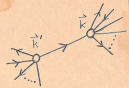

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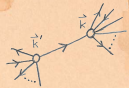

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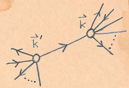

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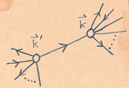

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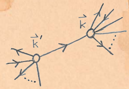
Global spreading condition:

Local growth equation:

- Define number of infected edges leading to nodes a distance d away from the original seed as $f(d)$.
- Infected edge growth equation:

$$f(d+1) = \mathbf{R} f(d).$$

- Applies for discrete time and continuous time contagion processes.
- Now see $B_{k_u,1}$ is the probability that an infected edge eventually infects a node.
- Also allows for recovery of nodes (SIR).



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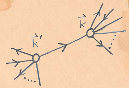

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- Define number of infected edges leading to nodes a distance d away from the original seed as $f(d)$.
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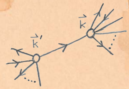

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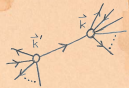

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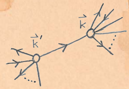

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Mixed, uncorrelated random netwoks:

- Now have two types of edges spreading infection: directed and undirected.
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  1. Infected directed edges can lead to infected directed or undirected edges.
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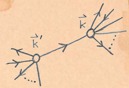

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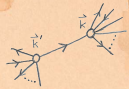

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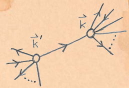

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- Two separate gain equations:

- Gain ratio matrix:

$$\mathbf{R} = \sum_{\vec{k}} \left[\begin{array}{cc} \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet (k_{\mathrm{u}} - 1) & \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{u}} \\ \frac{k_{\mathrm{u}} P_{\vec{k}}}{\langle k_{\mathrm{u}} \rangle} \bullet k_{\mathrm{o}} & \frac{k_{\mathrm{i}} P_{\vec{k}}}{\langle k_{\mathrm{i}} \rangle} \bullet k_{\mathrm{o}} \end{array}\right] \bullet B_{k_{\mathrm{u}}+k_{\mathrm{i}},1}$$

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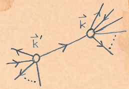

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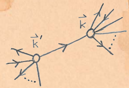

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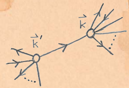

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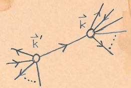

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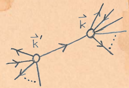

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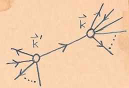

Correlated version:

- Now have to think of transfer of infection from edges emanating from degree $\vec{k}'$ nodes to edges emanating from degree $\vec{k}$ nodes.
- Replace $P^{(i)}(\vec{k} \mid *)$ with $P^{(i)}(\vec{k} \mid \vec{k}')$ and so on.
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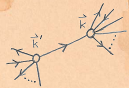

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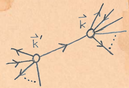

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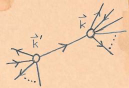

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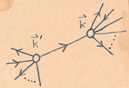

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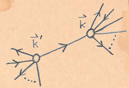
- V. Directed, Correlated—$f_{k_i k_o}(d+1) = \sum_{k'_i, k'_o} R_{k_i k_o k'_i k'_o} f_{k'_i k'_o}(d)$

$$R_{k_i k_o k'_i k'_o} = P^{(i)}(k_i, k_o \mid k'_i, k'_o) \bullet k_o \bullet B_{k_i k_o k'_i k'_o}$$

- VI. Mixed Directed and Undirected, Correlated—

$$\begin{bmatrix} f^{(u)}_{\vec{k}}(d+1) \\ f^{(o)}_{\vec{k}}(d+1) \end{bmatrix} = \sum_{\vec{k}'} \mathbf{R}_{\vec{k}\vec{k}'} \begin{bmatrix} f^{(u)}_{\vec{k}'}(d) \\ f^{(o)}_{\vec{k}'}(d) \end{bmatrix}$$

$$\mathbf{R}_{\vec{k}\vec{k}'} = \begin{bmatrix} P^{(u)}(\vec{k} \mid \vec{k}') \bullet (k_u - 1) & P^{(i)}(\vec{k} \mid \vec{k}') \bullet k_u \\ P^{(u)}(\vec{k} \mid \vec{k}') \bullet k_o & P^{(i)}(\vec{k} \mid \vec{k}') \bullet k_o \end{bmatrix} \bullet B_{\vec{k}\vec{k}'}$$



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$$R_{k_u k'_u} = P^{(u)}(k_u \mid k'_u) \bullet (k_u - 1) \bullet B_{k_u k'_u}$$

- V. Directed,
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- VI. Mixed Directed and Undirected, Correlated—

$$\begin{bmatrix} f^{(u)}_{\vec{k}}(d+1) \\ f^{(o)}_{\vec{k}}(d+1) \end{bmatrix} = \sum_{\vec{k}'} \mathbf{R}_{\vec{k}\vec{k}'} \begin{bmatrix} f^{(u)}_{\vec{k}'}(d) \\ f^{(o)}_{\vec{k}'}(d) \end{bmatrix}$$

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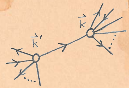

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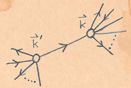

Summary of triggering probabilities for uncorrelated networks: [3]

- I. Undirected, Uncorrelated—
  $Q = \sum_{k_u'} P^{(u)}(k_u' \mid \cdot) B(1, k_u') \left[1 - (1 - Q)^{k_u' - 1}\right]$

$$S_{\mathrm{trig}} = \sum_{k_u'} P(k_u') \left[1 - (1 - Q)^{k_u'}\right]$$

- II. Directed, Uncorrelated—
  $Q = \sum_{k_i', k_o'} P^{(u)}(k_i', k_o' \mid \cdot) B(1, k_i') \left[1 - (1 - Q)^{k_o'}\right]$

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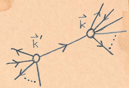

Summary of triggering probabilities for uncorrelated networks: [3]

- I. Undirected, Uncorrelated—
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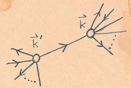


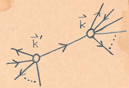
Summary of triggering probabilities for uncorrelated networks:

- III. Mixed Directed and Undirected, Uncorrelated—

$$Q^{(u)} = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}' \,|\, \cdot) B(1, \vec{k}') \left[1 - (1 - Q^{(u)})^{k'_{\mathrm{u}} - 1} (1 - Q^{(o)})^{k'_{\mathrm{o}}} \right]$$

$$Q^{(o)} = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}' \,|\, \cdot) B(1, \vec{k}') \left[1 - (1 - Q^{(u)})^{k'_{\mathrm{u}}} (1 - Q^{(o)})^{k'_{\mathrm{o}}} \right]$$

$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}') \left[1 - (1 - Q^{(u)})^{k'_{\mathrm{u}}} (1 - Q^{(o)})^{k'_{\mathrm{o}}} \right]$$



Summary of triggering probabilities for correlated networks:

- IV. Undirected, Correlated—

$Q_{k_u} = \sum_{k'_u} P^{(u)}(k'_u \mid k_u) B(1, k'_u) \left[1 - (1 - Q_{k'_u})^{k'_u - 1}\right]$

$$S_{\rm trig} = \sum_{k'_u} P(k'_u) \left[1 - (1 - Q_{k'_u})^{k'_u}\right]$$

- V. Directed, Correlated—

$Q_{k_i k_o} = \sum_{k'_i, k'_o} P^{(u)}(k'_i, k'_o \mid k_i, k_o) B(1, k'_i) \left[1 - (1 - Q_{k'_i k'_o})^{k'_o}\right]$

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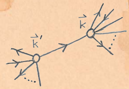

Summary of triggering probabilities for correlated networks:

- IV. Undirected, Correlated—
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  $Q_{k_{\mathrm{i}} k_{\mathrm{o}}} = \sum_{k'_{\mathrm{i}}, k'_{\mathrm{o}}} P^{(\mathrm{u})}(k'_{\mathrm{i}}, k'_{\mathrm{o}} \,|\, k_{\mathrm{i}}, k_{\mathrm{o}}) B(1, k'_{\mathrm{i}}) \left[1 - (1 - Q_{k'_{\mathrm{i}} k'_{\mathrm{o}}})^{k'_{\mathrm{o}}}\right]$

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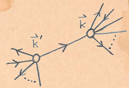

Summary of triggering probabilities for correlated networks:

- VI. Mixed Directed and Undirected, Correlated—

$$Q^{(u)}_{\vec{k}} = \sum_{\vec{k}} P^{(\mathrm{u})}(\vec{k}' \,|\, \vec{k}) B(1, \vec{k}') \left[1 - (1 - Q^{(u)}_{\vec{k}'})^{k'_{\mathrm{u}} - 1} (1 - Q^{(o)}_{\vec{k}'})^{k'_{\mathrm{o}}} \right]$$

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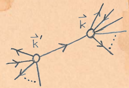

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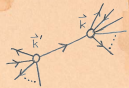
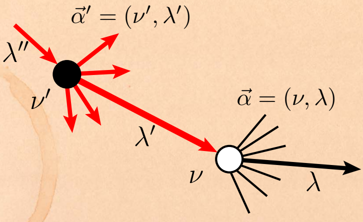
Outline



Full generalization:



$$f_{\vec{\alpha}}(d+1) = \sum_{\vec{\alpha}'} R_{\vec{\alpha}\vec{\alpha}'} f_{\vec{\alpha}'}(d)$$

$R_{\vec{\alpha}\vec{\alpha}'}$ is the gain ratio matrix and has the form:

$$R_{\vec{\alpha}\vec{\alpha}'} = P_{\vec{\alpha}\vec{\alpha}'} \bullet k_{\vec{\alpha}\vec{\alpha}'} \bullet B_{\vec{\alpha}\vec{\alpha}'}.$$

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$$\max |\mu| : \mu \in \sigma(\mathbf{R}) > 1$$

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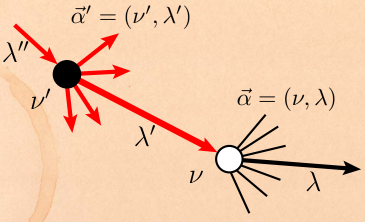

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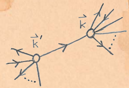

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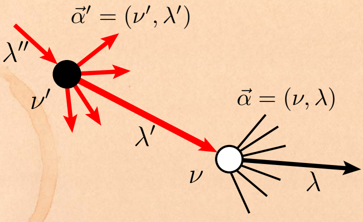

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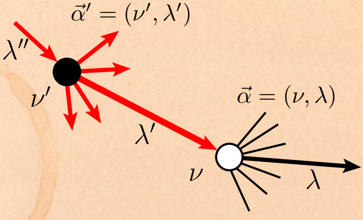

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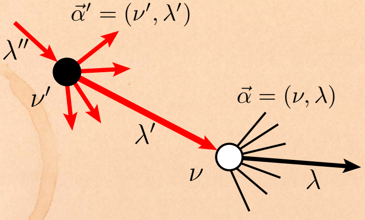

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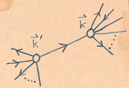
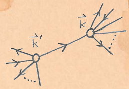


Nutshell:

- Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.
- Spreading conditions and triggering probabilities of contagion processes can be determined using a direct, physical approach.
- These conditions can be generalized to arbitrary random networks with arbitrary node and edge types.



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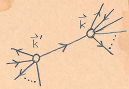

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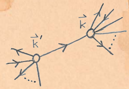

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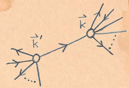

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