Mixed, correlated random networks

Complex Networks
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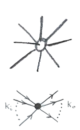
Outline

Random directed networks:



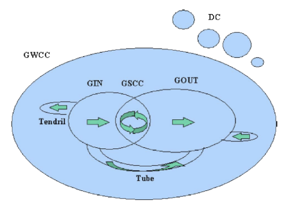
- So far, we've studied networks with undirected, unweighted edges.
- Now consider directed, unweighted edges.
- Nodes have k_i and k_o incoming and outgoing edges, otherwise random.
- Network defined by joint in- and out-degree distribution: P_{k_i,k_o}
- Normalization: $\sum_{k_i=0}^{\infty}\sum_{k_o=0}^{\infty} P_{k_i,k_o} = 1$
- Marginal in-degree and out-degree distributions:

$$P_{k_i} = \sum_{k_o=0}^{\infty} P_{k_i,k_o} \text{ and } P_{k_o} = \sum_{k_i=0}^{\infty} P_{k_i,k_o}$$

- Required balance:

$$\langle k_i \rangle = \sum_{k_i=0}^{\infty}\sum_{k_o=0}^{\infty} k_i P_{k_i,k_o} = \sum_{k_i=0}^{\infty}\sum_{k_o=0}^{\infty} k_o P_{k_i,k_o} = \langle k_o \rangle$$

Directed network structure:



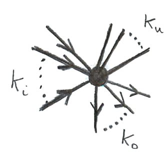
From Boguñá and Serano. [1]

- GWCC = Giant Weakly Connected Component (directions removed);
- GIN = Giant In-Component;
- GOUT = Giant Out-Component;
- GSCC = Giant Strongly Connected Component;
- DC = Disconnected Components (finite).
- When moving through a family of increasingly connected directed random networks, GWCC usually appears before GIN, GOUT, and GSCC which tend to appear together. [4, 1]

Mixed random networks:

Important observation:

- Directed and undirected random networks are separate families...
- ...and analyses are also disjoint.
- Need to examine a larger family of random networks with mixed directed and undirected edges.



- Consider nodes with three types of edges:
  1. k_u undirected edges,
  2. k_i incoming directed edges,
  3. k_o outgoing directed edges.
- Define a node by generalized degree:

$$\vec{k} = [\, k_u \;\; k_i \;\; k_o \,]^{\mathrm{T}}.$$

Mixed random networks:

- Joint degree distribution:

$$P_{\vec{k}} \text{ where } \vec{k} = [\, k_u \;\; k_i \;\; k_o \,]^{\mathrm{T}}.$$

- As for directed networks, require in- and out-degree averages to match up:

$$\langle k_i \rangle = \sum_{k_u=0}^{\infty}\sum_{k_i=0}^{\infty}\sum_{k_o=0}^{\infty} k_i P_{\vec{k}} = \sum_{k_u=0}^{\infty}\sum_{k_i=0}^{\infty}\sum_{k_o=0}^{\infty} k_o P_{\vec{k}} = \langle k_o \rangle$$

- Otherwise, no other restrictions and connections are random.
- Directed and undirected random networks are disjoint subfamilies:

$$\text{Undirected: } P_{\vec{k}} = P_{k_u}\delta_{k_i,0}\delta_{k_o,0},$$

$$\text{Directed: } P_{\vec{k}} = \delta_{k_u,0}P_{k_i,k_o}.$$

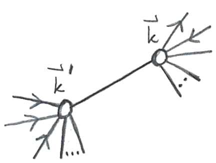
Correlations:

- Now add correlations (two point or Markovian):
  1. $P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')$ = probability that an undirected edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node.
  2. $P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an in-directed edge relative to the destination node.
  3. $P^{(\mathrm{o})}(\vec{k}\,|\,\vec{k}')$ = probability that an edge leaving a degree $\vec{k}'$ nodes arrives at a degree $\vec{k}$ node is an out-directed edge relative to the destination node.
- Now require more refined (detailed) balance.
- Conditional probabilities cannot be arbitrary.
  1. $P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')$ must be related to $P^{(\mathrm{u})}(\vec{k}'\,|\,\vec{k})$.
  2. $P^{(\mathrm{o})}(\vec{k}\,|\,\vec{k}')$ and $P^{(\mathrm{o})}(\vec{k}\,|\,\vec{k}')$ must be connected.

Correlations—Undirected edge balance:

Mixed, correlated random networks | Directed random networks | Mixed random networks | Definition | Correlations | Mixed Random Network Contagion | Spreading condition | Full generalization | Nutshell | References

- Randomly choose an edge, and randomly choose one end.
- Say we find a degree $\vec{k}$ node at this end, and a degree $\vec{k}'$ node at the other end.
- Define probability this happens as $P^{(\mathrm{u})}(\vec{k},\vec{k}')$.
- Observe we must have $P^{(\mathrm{u})}(\vec{k},\vec{k}') = P^{(\mathrm{u})}(\vec{k}',\vec{k})$.



- Conditional probability connection:

$$\begin{aligned} P^{(\mathrm{u})}(\vec{k},\vec{k}') &= P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}')\frac{k_u' P(\vec{k}')}{\langle k_u' \rangle} \\ &\ \ ||| \\ P^{(\mathrm{u})}(\vec{k}',\vec{k}) &= P^{(\mathrm{u})}(\vec{k}'\,|\,\vec{k})\frac{k_u P(\vec{k})}{\langle k_u \rangle}. \end{aligned}$$

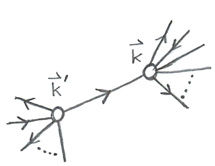
Correlations—Directed edge balance:

Mixed, correlated random networks | Directed random networks | Mixed random networks | Definition | Correlations | Mixed Random Network Contagion | Spreading condition | Full generalization | Nutshell | References

- The quantities

$$\frac{k_o P(\vec{k})}{\langle k_o \rangle} \text{ and } \frac{k_i P(\vec{k})}{\langle k_i \rangle}$$

give the probabilities that in starting at a random end of a randomly selected edge, we begin at a degree $\vec{k}$ node and then find ourselves travelling:



  1. along an outgoing edge, or
  2. against the direction of an incoming edge.
- We therefore have

$$P^{(\mathrm{dir})}(\vec{k},\vec{k}') = P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}')\frac{k_o' P(\vec{k}')}{\langle k_o' \rangle} = P^{(\mathrm{o})}(\vec{k}'\,|\,\vec{k})\frac{k_i P(\vec{k})}{\langle k_i \rangle}.$$

- Note that $P^{(\mathrm{dir})}(\vec{k},\vec{k}')$ and $P^{(\mathrm{dir})}(\vec{k}',\vec{k})$ are in general not related if $\vec{k} \neq \vec{k}'$.

Global spreading condition: [2]

Mixed, correlated random networks | Directed random networks | Mixed random networks | Definition | Correlations | Mixed Random Network Contagion | Spreading condition | Full generalization | Nutshell | References

When are cascades possible?:

- Consider uncorrelated mixed networks first.
- Recall our first result for undirected random networks, that edge gain ratio must exceed 1:

$$\mathbf{R} = \sum_{k_u=0}^{\infty} \frac{k_u P_{k_u}}{\langle k_u \rangle} \bullet (k_u - 1) \bullet B_{k_u,1} > 1.$$

- Similar form for purely directed networks:

$$\mathbf{R} = \sum_{k_i=0}^{\infty}\sum_{k_o=0}^{\infty} \frac{k_i P_{k_i,k_o}}{\langle k_i \rangle} \bullet k_o \bullet B_{k_i,1} > 1.$$

- Both are composed of (1) probability of connection to a node of a given type; (2) number of newly infected edges if successful; and (3) probability of infection.

Global spreading condition:

Mixed, correlated random networks | Directed random networks | Mixed random networks | Definition | Correlations | Mixed Random Network Contagion | Spreading condition | Full generalization | Nutshell | References

Local growth equation:

- Define number of infected edges leading to nodes a distance d away from the original seed as $f(d)$.
- Infected edge growth equation:

$$f(d+1) = \mathbf{R}f(d).$$

- Applies for discrete time and continuous time contagion processes.
- Now see $B_{k_u,1}$ is the probability that an infected edge eventually infects a node.
- Also allows for recovery of nodes (SIR).

Global spreading condition:

Mixed, correlated random networks | Directed random networks | Mixed random networks | Definition | Correlations | Mixed Random Network Contagion | Spreading condition | Full generalization | Nutshell | References

Mixed, uncorrelated random netwoks:

- Now have two types of edges spreading infection: directed and undirected.
- Gain ratio now more complicated:
  1. Infected directed edges can lead to infected directed or undirected edges.
  2. Infected undirected edges can lead to infected directed or undirected edges.
- Define $f^{(\mathrm{u})}(d)$ and $f^{(\mathrm{o})}(d)$ as the expected number of infected undirected and directed edges leading to nodes a distance d from seed.

- Gain ratio now has a matrix form:

$$\begin{bmatrix} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{bmatrix}$$

- Two separate gain equations:

$$f^{(\mathrm{u})}(d+1) = \frac{k_\mathrm{u} P_{\vec{k}}}{\langle k_\mathrm{u} \rangle} \bullet (k_\mathrm{u}-1) B_{k_\mathrm{u}+k_\mathrm{i},1} f^{(\mathrm{u})}(d) + \frac{k_\mathrm{i} P_{\vec{k}}}{\langle k_\mathrm{i} \rangle} \bullet k_\mathrm{u} \bullet B_{k_\mathrm{u}+k_\mathrm{i},1} f^{(\mathrm{o})}(d)$$

$$f^{(\mathrm{o})}(d+1) = \frac{k_\mathrm{u} P_{\vec{k}}}{\langle k_\mathrm{u} \rangle} \bullet k_\mathrm{o} B_{k_\mathrm{u}+k_\mathrm{i},1} f^{(\mathrm{u})}(d) + \frac{k_\mathrm{i} P_{\vec{k}}}{\langle k_\mathrm{i} \rangle} \bullet k_\mathrm{o} \bullet B_{k_\mathrm{u}+k_\mathrm{i},1} f^{(\mathrm{o})}(d)$$

- Gain ratio matrix:

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} \frac{k_\mathrm{u} P_{\vec{k}}}{\langle k_\mathrm{u} \rangle} \bullet (k_\mathrm{u}-1) & \frac{k_\mathrm{i} P_{\vec{k}}}{\langle k_\mathrm{i} \rangle} \bullet k_\mathrm{u} \\ \frac{k_\mathrm{u} P_{\vec{k}}}{\langle k_\mathrm{u} \rangle} \bullet k_\mathrm{o} & \frac{k_\mathrm{i} P_{\vec{k}}}{\langle k_\mathrm{i} \rangle} \bullet k_\mathrm{o} \end{bmatrix} \bullet B_{k_\mathrm{u}+k_\mathrm{i},1}$$

- Spreading condition: max eigenvalue of $\mathbf{R} > 1$.

Global spreading condition:

- Useful change of notation for making results more general: write $P^{(\mathrm{u})}(\vec{k}\,|\,*) = \frac{k_\mathrm{u} P_{\vec{k}}}{\langle k_\mathrm{u} \rangle}$ and $P^{(\mathrm{i})}(\vec{k}\,|\,*) = \frac{k_\mathrm{i} P_{\vec{k}}}{\langle k_\mathrm{i} \rangle}$ where $*$ indicates the starting node's degree is irrelevant (no correlations).
- Also write $B_{k_\mathrm{u} k_\mathrm{i},*}$ to indicate a more general infection probability, but one that does not depend on the edge's origin.
- Now have, for the example of mixed, uncorrelated random networks:

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet (k_\mathrm{u}-1) & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_\mathrm{u} \\ P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet k_\mathrm{o} & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_\mathrm{o} \end{bmatrix} \bullet B_{k_\mathrm{u} k_\mathrm{i},*}$$

Summary of contagion conditions for uncorrelated networks:

- I. Undirected, Uncorrelated—$f(d+1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_\mathrm{u}} P^{(\mathrm{u})}(k_\mathrm{u}\,|\,*) \bullet (k_\mathrm{u}-1) \bullet B_{k_\mathrm{u},*}$$

- II. Directed, Uncorrelated—$f(d+1) = \mathbf{f}(d)$:

$$\mathbf{R} = \sum_{k_\mathrm{i},k_\mathrm{o}} P^{(\mathrm{i})}(k_\mathrm{i}, k_\mathrm{o}\,|\,*) \bullet k_\mathrm{o} \bullet B_{k_\mathrm{i},*}$$

- III. Mixed Directed and Undirected, Uncorrelated—

$$\begin{bmatrix} f^{(\mathrm{u})}(d+1) \\ f^{(\mathrm{o})}(d+1) \end{bmatrix} = \mathbf{R} \begin{bmatrix} f^{(\mathrm{u})}(d) \\ f^{(\mathrm{o})}(d) \end{bmatrix}$$

$$\mathbf{R} = \sum_{\vec{k}} \begin{bmatrix} P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet (k_\mathrm{u}-1) & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_\mathrm{u} \\ P^{(\mathrm{u})}(\vec{k}\,|\,*) \bullet k_\mathrm{o} & P^{(\mathrm{i})}(\vec{k}\,|\,*) \bullet k_\mathrm{o} \end{bmatrix} \bullet B_{k_\mathrm{u} k_\mathrm{i},*}$$

Correlated version:

- Now have to think of transfer of infection from edges emanating from degree $\vec{k}'$ nodes to edges emanating from degree $\vec{k}$ nodes.
- Replace $P^{(\mathrm{i})}(\vec{k}\,|\,*)$ with $P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}')$ and so on.
- Edge types are now more diverse beyond directed and undirected as originating node type matters.
- Sums are now over $\vec{k}'$.

Summary of contagion conditions for correlated networks:

- IV. Undirected, Correlated—$f_{k_\mathrm{u}}(d+1) = \sum_{k'_\mathrm{u}} R_{k_\mathrm{u} k'_\mathrm{u}} f_{k'_\mathrm{u}}(d)$

$$R_{k_\mathrm{u} k'_\mathrm{u}} = P^{(\mathrm{u})}(k_\mathrm{u}\,|\,k'_\mathrm{u}) \bullet (k_\mathrm{u}-1) \bullet B_{k_\mathrm{u} k'_\mathrm{u}}$$

- V. Directed, Correlated—$f_{k_\mathrm{i} k_\mathrm{o}}(d+1) = \sum_{k'_\mathrm{i},k'_\mathrm{o}} R_{k_\mathrm{i} k_\mathrm{o} k'_\mathrm{i} k'_\mathrm{o}} f_{k'_\mathrm{i} k'_\mathrm{o}}(d)$

$$R_{k_\mathrm{i} k_\mathrm{o} k'_\mathrm{i} k'_\mathrm{o}} = P^{(\mathrm{i})}(k_\mathrm{i}, k_\mathrm{o}\,|\,k'_\mathrm{i}, k'_\mathrm{o}) \bullet k_\mathrm{o} \bullet B_{k_\mathrm{i} k_\mathrm{o} k'_\mathrm{i} k'_\mathrm{o}}$$

- VI. Mixed Directed and Undirected, Correlated—

$$\begin{bmatrix} f^{(\mathrm{u})}_{\vec{k}}(d+1) \\ f^{(\mathrm{o})}_{\vec{k}}(d+1) \end{bmatrix} = \sum_{\vec{k}'} \mathbf{R}_{\vec{k}\vec{k}'} \begin{bmatrix} f^{(\mathrm{u})}_{\vec{k}'}(d) \\ f^{(\mathrm{o})}_{\vec{k}'}(d) \end{bmatrix}$$

$$\mathbf{R}_{\vec{k}\vec{k}'} = \begin{bmatrix} P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}') \bullet (k_\mathrm{u}-1) & P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}') \bullet k_\mathrm{u} \\ P^{(\mathrm{u})}(\vec{k}\,|\,\vec{k}') \bullet k_\mathrm{o} & P^{(\mathrm{i})}(\vec{k}\,|\,\vec{k}') \bullet k_\mathrm{o} \end{bmatrix} \bullet B_{\vec{k}\vec{k}'}$$

Summary of triggering probabilities for uncorrelated networks: [3]

- I. Undirected, Uncorrelated—
$Q = \sum_{k'_\mathrm{u}} P^{(\mathrm{u})}(k'_\mathrm{u}\,|\,\cdot) B(1, k'_\mathrm{u}) \left[1 - (1-Q)^{k'_\mathrm{u}-1}\right]$

$$S_\mathrm{trig} = \sum_{k'_\mathrm{u}} P(k'_\mathrm{u}) \left[1 - (1-Q)^{k'_\mathrm{u}}\right]$$

- II. Directed, Uncorrelated—
$Q = \sum_{k'_\mathrm{i},k'_\mathrm{o}} P^{(\mathrm{u})}(k'_\mathrm{i}, k'_\mathrm{o}\,|\,\cdot) B(1, k'_\mathrm{i}) \left[1 - (1-Q)^{k'_\mathrm{o}}\right]$

$$S_\mathrm{trig} = \sum_{k'_\mathrm{i},k'_\mathrm{o}} P(k'_\mathrm{i}, k'_\mathrm{o}) \left[1 - (1-Q)^{k'_\mathrm{o}}\right]$$

Summary of triggering probabilities for uncorrelated networks:

- III. Mixed Directed and Undirected, Uncorrelated—

$$Q^{(u)} = \sum_{\vec{k}'} P^{(\mathrm{u})}(\vec{k}'\,|\,\cdot)B(1,\vec{k}')\left[1-(1-Q^{(u)})^{k'_{\mathrm{u}}-1}(1-Q^{(o)})^{k'_{\mathrm{o}}}\right]$$

$$Q^{(o)} = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}'\,|\,\cdot)B(1,\vec{k}')\left[1-(1-Q^{(u)})^{k'_{\mathrm{u}}}(1-Q^{(o)})^{k'_{\mathrm{o}}}\right]$$

$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}')\left[1-(1-Q^{(u)})^{k'_{\mathrm{u}}}(1-Q^{(o)})^{k'_{\mathrm{o}}}\right]$$

Summary of triggering probabilities for correlated networks:

- IV. Undirected, Correlated—
$Q_{k_{\mathrm{u}}} = \sum_{k'_{\mathrm{u}}} P^{(\mathrm{u})}(k'_{\mathrm{u}}\,|\,k_{\mathrm{u}})B(1,k'_{\mathrm{u}})\left[1-(1-Q_{k'_{\mathrm{u}}})^{k'_{\mathrm{u}}-1}\right]$

$$S_{\mathrm{trig}} = \sum_{k'_{\mathrm{u}}} P(k'_{\mathrm{u}})\left[1-(1-Q_{k'_{\mathrm{u}}})^{k'_{\mathrm{u}}}\right]$$

- V. Directed, Correlated—
$Q_{k_{\mathrm{i}}k_{\mathrm{o}}} = \sum_{k'_{\mathrm{i}},k'_{\mathrm{o}}} P^{(\mathrm{u})}(k'_{\mathrm{i}},k'_{\mathrm{o}}\,|\,k_{\mathrm{i}},k_{\mathrm{o}})B(1,k'_{\mathrm{i}})\left[1-(1-Q_{k'_{\mathrm{i}}k'_{\mathrm{o}}})^{k'_{\mathrm{o}}}\right]$

$$S_{\mathrm{trig}} = \sum_{k'_{\mathrm{i}},k'_{\mathrm{o}}} P(k'_{\mathrm{i}},k'_{\mathrm{o}})\left[1-(1-Q_{k'_{\mathrm{i}}k'_{\mathrm{o}}})^{k'_{\mathrm{o}}}\right]$$

Summary of triggering probabilities for correlated networks:

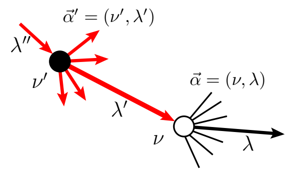
- VI. Mixed Directed and Undirected, Correlated—

$$Q^{(u)}_{\vec{k}} = \sum_{\vec{k}} P^{(\mathrm{u})}(\vec{k}'\,|\,\vec{k})B(1,\vec{k}')\left[1-(1-Q^{(u)}_{\vec{k}'})^{k'_{\mathrm{u}}-1}(1-Q^{(o)}_{\vec{k}'})^{k'_{\mathrm{o}}}\right]$$

$$Q^{(o)}_{\vec{k}} = \sum_{\vec{k}'} P^{(\mathrm{i})}(\vec{k}'\,|\,\vec{k})B(1,\vec{k}')\left[1-(1-Q^{(u)}_{\vec{k}'})^{k'_{\mathrm{u}}}(1-Q^{(o)}_{\vec{k}'})^{k'_{\mathrm{o}}}\right]$$

$$S_{\mathrm{trig}} = \sum_{\vec{k}'} P(\vec{k}')\left[1-(1-Q^{(u)}_{\vec{k}'})^{k'_{\mathrm{u}}}(1-Q^{(o)}_{\vec{k}'})^{k'_{\mathrm{o}}}\right]$$

Full generalization:



$$f_{\vec{\alpha}}(d+1) = \sum_{\vec{\alpha}'} R_{\vec{\alpha}\vec{\alpha}'} f_{\vec{\alpha}'}(d)$$

$R_{\vec{\alpha}\vec{\alpha}'}$ is the gain ratio matrix and has the form:

$$R_{\vec{\alpha}\vec{\alpha}'} = P_{\vec{\alpha}\vec{\alpha}'} \bullet k_{\vec{\alpha}\vec{\alpha}'} \bullet B_{\vec{\alpha}\vec{\alpha}'}.$$

- $P_{\vec{\alpha}\vec{\alpha}'}$ = conditional probability that a type λ' edge emanating from a type ν' node leads to a type ν node.
- $k_{\vec{\alpha}\vec{\alpha}'}$ = potential number of newly infected edges of type λ emanating from nodes of type ν.
- $B_{\vec{\alpha}\vec{\alpha}'}$ = probability that a type ν node is eventually infected by a single infected type λ' link arriving from a neighboring node of type ν'.
- Generalized contagion condition:

$$\max|\mu| : \mu \in \sigma(\mathbf{R}) > 1$$

Nutshell:

- Mixed, correlated random networks with undirected and directed edges form natural inclusive generalization of purely undirected and purely directed random networks.
- Spreading conditions and triggering probabilities of contagion processes can be determined using a direct, physical approach.
- These conditions can be generalized to arbitrary random networks with arbitrary node and edge types.

References I

[1] M. Boguñá and M. Ángeles Serrano.
Generalized percolation in random directed networks.
Phys. Rev. E, 72:016106, 2005. pdf (⊞)

[2] P. S. Dodds, K. D. Harris, and J. L. Payne.
Direct, phyiscally-motivated derivation of the contagion condition for spreading processes on generalized random networks.
http://arxiv.org/abs/1101.5591, 2011.
pdf (⊞)

[3] P. S. Dodds, K. D. Harris, and J. L. Payne.
Direct, physically-motivated derivation of triggering probabilities for contagion processes acting on correlated random networks.
2011.

References II

[4] M. E. J. Newman, S. H. Strogatz, and D. J. Watts.
Random graphs with arbitrary degree distributions and their applications.
Phys. Rev. E, 64:026118, 2001. pdf (⊞)