Diffusion

Random walks on networks

Random walks and diffusion on networks

Complex Networks CSYS/MATH 303, Spring, 2011

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Outline

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Diffusion

- Imagine a single random walker moving around on a network.
- Q: What's the long term probability distribution for where the walker will be?
- Define p_i(t) as the probability that at time step t, our walker is at node t.
- > We want to characterize the evolution of $\vec{p}(t)$
- First task, connect $\vec{p}(t+1)$ to $\vec{p}(t)$.



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- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.

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- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Worse still: Barry is hopelessly drunk.

Diffusion

Random walks on networks

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 Consider simple undirected, ergodic (strongly connected) networks.

Barry is at node / at time t with probability p_j(t).
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Barry arrives at node / from node / with probability
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Barry is at node *j* at time *t* with probability *p_j(t)*.
 Induction of the probability *p_j(t)*.



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- Equation-wise:

$$p_i(t+1) = \sum_{j=1}^n \frac{1}{k_j} a_{ji} p_j(t).$$

where k_i is j's degree.





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where k_j is j's degree. Note: $k_i = \sum_{j=1}^n a_{ij}$.

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Random walks on networks

Excellent observation: The same equation applies for stuff moving around a network, such that at each time step all material at node *i* is sent to its neighbors.

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diffusion (\boxplus)



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► Random walking is equivalent to diffusion (⊞).



Linear algebra-based excitement: $p_i(t+1) = \sum_{j=1}^n a_{jj} \frac{1}{k_j} p_j(t)$ is more usefully viewed as

 $\vec{p}(t+1) = A^{\mathrm{T}} K^{-1} \vec{p}(t)$

where $[K_{ij}] = [\delta_{ij}k_i]$ has node degrees on the main diagonal and zeros everywhere else.

Expect this eigenvalue will be 1 (doesn't make sens for total probability to change).

 The corresponding eigenvector will be the limitin probability distribution (or invariant measure).
 Extra concerns: multiplicity of eigenvalue = 1; an network condectedness.





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- Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.



By inspection, we see that

$$\vec{p}(\infty) = \frac{1}{\sum_{i=1}^{n} k_i} \vec{k}$$

satisfies $\vec{p}(\infty) = A^{T} K^{-1} \vec{p}(\infty)$ with eigenvalue 1.

First proportional to its degree A.
Nice implication: probability of finding Barry travelling along-any edge is uniform
Diffusion in real space smooths things out.
On networks, uniformity occurs on edges.
So in fact, diffusion in real space is about the edges too but we just don't see that.





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• Goodness: $A^{T}K^{-1}$ is similar to a real symmetric matrix if $A = A^{T}$.

 $K^{-1/2}A^{\mathrm{T}}K^{-1}K^{1/2} = K^{-1/2}A^{\mathrm{T}}K^{-1/2}$

Since $A^{T} = A$, we have

 $(K^{-1/2}AK^{-1/2})^{\mathrm{T}} = K^{-1/2}AK^{-1/2}$

Upshot: A¹K⁻¹ = A^{K-1} has real eigenvalues and a complete set of orthogonal eigenvectors.
 Can also show that maximum eigenvalue magnitude is indeed 1.



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