# Random walks and diffusion on networks

## Complex Networks
## CSYS/MATH 303, Spring, 2011

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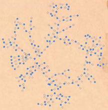
Vermont Advanced Computing Center
University of Vermont



# Outline

Random walks on networks

# Random walks on networks—basics:

- Imagine a single random walker moving around on a network.
- At $t = 0$, start walker at node $j$ and take time to be discrete.
- Q: What's the long term probability distribution for where the walker will be?
- Define $p_i(t)$ as the probability that at time step $t$, our walker is at node $i$.
- We want to characterize the evolution of $\vec{p}(t)$.
- First task: connect $\vec{p}(t + 1)$ to $\vec{p}(t)$.

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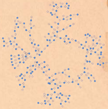

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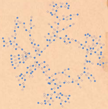

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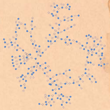
- Let's call our walker Barry.
- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Worse still: Barry is hopelessly drunk.



# Where is Barry?

- Consider simple undirected, ergodic (strongly connected) networks.
- As usual, represent network by adjacency matrix $A$ where

$$a_{ij} = 1 \text{ if } i \text{ has an edge leading to } j,$$
$$a_{ij} = 0 \text{ otherwise.}$$

- Barry is at node $j$ at time $t$ with probability $p_j(t)$.
- In the next time step, he randomly lurches toward one of $j$'s neighbors.
- Barry arrives at node $i$ from node $j$ with probability $\frac{1}{k_j}$ if an edge connects $j$ to $i$.
- Equation-wise:

$$p_i(t+1) = \sum_{j=1}^{n} \frac{1}{k_j} a_{ji} p_j(t).$$

where $k_j$ is $j$'s degree.

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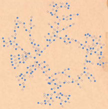

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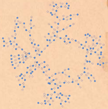

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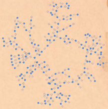

# Inebriation and diffusion:

- Excellent observation: The same equation applies for stuff moving around a network, such that at each time step all material at node *i* is sent to its neighbors.
- $x_i(t)$ = amount of stuff at node $i$ at time $t$.
- $$x_i(t+1) = \sum_{j=1}^{n} \frac{1}{k_j} a_{ji} x_j(t).$$
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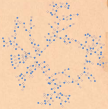

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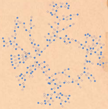

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- Linear algebra-based excitement:
  $p_i(t+1) = \sum_{j=1}^{n} a_{ji} \frac{1}{k_j} p_j(t)$ is more usefully viewed as

  $$\vec{p}(t+1) = A^{\mathrm{T}} K^{-1} \vec{p}(t)$$

  where $[K_{ij}] = [\delta_{ij} k_i]$ has node degrees on the main diagonal and zeros everywhere else.
- So... we need to find the dominant eigenvalue of $A^{\mathrm{T}} K^{-1}$.
- Expect this eigenvalue will be 1 (doesn't make sense for total probability to change).
- The corresponding eigenvector will be the limiting probability distribution (or invariant measure).
- Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.

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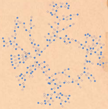

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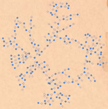

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- By inspection, we see that

$$\vec{p}(\infty) = \frac{1}{\sum_{i=1}^{n} k_i}\vec{k}$$

satisfies $\vec{p}(\infty) = A^{\mathrm{T}} K^{-1} \vec{p}(\infty)$ with eigenvalue 1.

- We will find Barry at node $i$ with probability proportional to its degree $k_i$.
- Nice implication: probability of finding Barry travelling along any edge is uniform.
- Diffusion in real space smooths things out.
- On networks, uniformity occurs on edges.
- So in fact, diffusion in real space is about the edges too but we just don't see that.

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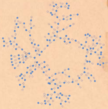

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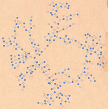

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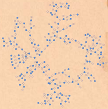

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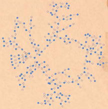

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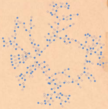

# Other pieces:

- Goodness: $A^{\mathrm{T}}K^{-1}$ is similar to a real symmetric matrix if $A = A^{\mathrm{T}}$.
- Consider the transformation $M = K^{-1/2}$:

$$K^{-1/2}A^{\mathrm{T}}K^{-1}K^{1/2} = K^{-1/2}A^{\mathrm{T}}K^{-1/2}.$$

- Since $A^{\mathrm{T}} = A$, we have

$$(K^{-1/2}AK^{-1/2})^{\mathrm{T}} = K^{-1/2}AK^{-1/2}.$$

- Upshot: $A^{\mathrm{T}}K^{-1} = AK^{-1}$ has real eigenvalues and a complete set of orthogonal eigenvectors.
- Can also show that maximum eigenvalue magnitude is indeed 1.

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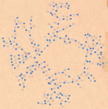

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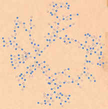

# Other pieces:

- Goodness: $A^{\mathrm{T}}K^{-1}$ is similar to a real symmetric matrix if $A = A^{\mathrm{T}}$.
- Consider the transformation $M = K^{-1/2}$:

$$K^{-1/2}A^{\mathrm{T}}K^{-1}K^{1/2} = K^{-1/2}A^{\mathrm{T}}K^{-1/2}.$$

- Since $A^{\mathrm{T}} = A$, we have

$$(K^{-1/2}AK^{-1/2})^{\mathrm{T}} = K^{-1/2}AK^{-1/2}.$$

- Upshot: $A^{\mathrm{T}}K^{-1} = AK^{-1}$ has real eigenvalues and a complete set of orthogonal eigenvectors.
- Can also show that maximum eigenvalue magnitude is indeed 1.