Branching Networks I

Complex Networks CSYS/MATH 303, Spring, 2011

Prof. Peter Dodds

Department of Mathematics & Statistics Center for Complex Systems Vermont Advanced Computing Center University of Vermont









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Introduction Definitions Allometry Laws Stream Ordering Horton's Laws Tokunaga's Law Nutshell References





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Branching networks are useful things:

- Fundamental to material supply and collection
 - Collection: From many sources to one sink in 2+ 3-d.
- Typically observe hierarchical, recursive self-simil structure

Examples:

- River networks (our focus)
- Cardiovascular networks
- Plants
- Evolutionary trees
- Organizations (only in theory...)

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Nutshell

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Branching networks are everywhere...



http://hydrosheds.cr.usgs.gov/ (田)

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Branching networks are everywhere...



http://en.wikipedia.org/wiki/Image:Applebox.JPG (III)

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A beautiful simulation of erosion:

Bruce Shaw (LDEO, Columbia) and Marcelo Magnasco (Rockefeller)

Branching Networks I





Definitions

- Drainage basin for a point p is the complete region of land from which overland flow drains through p.
 - Recursive structure: Basins contain basins and so
- In principle, a drainage basin is defined at every point on a landscape.
 - On Pathilslopes, drainage basins are effectively
- We treat subsurface and surface flow as following gradient of the surface.
- Okay for large-scale networks.

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- a = drainage
 basin area
- length of longest (main) stream (which may be fractal)
- L = L_{||} = longitudinal length of basin
- L = L_⊥ = width of basin

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 L_{\perp} L_1 a $L_{\parallel} = L$ L_{\perp}

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Allometry

► Isometry:

dimensions scale linearly with each other. Branching Networks I





Allometry

► Isometry:

dimensions scale linearly with each other. Allometry: dimensions scale nonlinearly.

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Allometric relationships:

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Allometric relationships:





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Allometric relationships:



 $\ell \propto L^d$

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Allometric relationships:

 $\ell \propto L^d$

 $\ell \propto a^h$

Combine above:

 $\pmb{a} \propto \pmb{L}^{d/h} \equiv \pmb{L}^D$

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Hack's law (1957)^[2]:

reportedly 0.5 < h < 0.7

 $\ell \propto a^h$

basins elonga

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Hack's law (1957)^[2]:

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Scaling of main stream length with basin size:

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Scaling of main stream length with basin size:

 $\ell \propto L_{\parallel}^d$

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Basin allometry:

 $L_{\parallel} \propto a^{h/d} \equiv a^{1/D}$

 $D < 2 \rightarrow$ basins elongate.

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There are a few more 'laws':^[1]

Relation: Name or description:

 $T_k = T_1 (R_T)^k$ Tokunaga's law $\ell \sim I^d$ self-affinity of single channels Horton's law of stream numbers $n_{\omega}/n_{\omega+1}=R_n$ $\bar{\ell}_{\omega+1}/\bar{\ell}_{\omega} = R_{\ell}$ Horton's law of main stream lengths $\bar{a}_{\omega+1}/\bar{a}_{\omega}=R_a$ Horton's law of basin areas $\bar{s}_{\omega+1}/\bar{s}_{\omega}=R_s$ Horton's law of stream segment lengths $L_{\perp} \sim L^{H}$ scaling of basin widths $P(a) \sim a^{-\tau}$ probability of basin areas $P(\ell) \sim \ell^{-\gamma}$ probability of stream lengths $\ell \sim a^h$ Hack's law $a \sim L^D$ scaling of basin areas $\Lambda \sim a^{\beta}$ Langbein's law $\lambda \sim I^{\varphi}$ variation of Langbein's law

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Reported parameter values:^[1]

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arameter:	Real networks
R _n	3.0–5.0
Ra	3.0-6.0
$R_\ell = R_T$	1.5–3.0
<i>T</i> ₁	1.0–1.5
d	1.1 ± 0.01
D	1.8 ± 0.1
h	0.50-0.70
au	$\textbf{1.43} \pm \textbf{0.05}$
γ	1.8 ± 0.1
Н	0.75–0.80
β	0.50-0.70
arphi	1.05 ± 0.05

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Order of business:

Find out how these relationships are connected
 Determine most fundamental description.
 Explain origins of these parameter values

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Order of business:

1. Find out how these relationships are connected.

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Order of business:

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Order of business:

- 1. Find out how these relationships are connected.
- 2. Determine most fundamental description.
- 3. Explain origins of these parameter values

For (3): Many attempts: not yet sorted out...

Branching Networks





Method for describing network architecture:

- Introduced by Horton (1945) ^[3]
- ► Modified by Stranler (1957)¹⁰
- Term: Horton-Strahler Stream Ordering^[4]
 - Can be seen as iterative trimming of a network

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Some definitions:

- A channel head is a point in landscape where flow becomes focused enough to form a stream.
 - reaches from a channel head to a junction with another stream.
 - Hertighty analogous to capillary vessels.

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- A source stream is defined as the stream that reaches from a channel head to a junction with another stream.

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Nutshell





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- A channel head is a point in landscape where flow becomes focused enough to form a stream.
- A source stream is defined as the stream that reaches from a channel head to a junction with another stream.
- Roughly analogous to capillary vessels.
- Use symbol $\omega = 1, 2, 3, ...$ for stream order.

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1. Label all source streams as order $\omega = 1$ and remo 2. Later streams as order $\omega = 2$ and remove.

Repeat until one stream is left (order = Ω)
 Basin is said to be of the order of the last stream removed.

5. Example above is a basin of order to

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- 2. Label all new source streams as order $\omega = 2$ and remove.

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- 2. Label all new source streams as order $\omega = 2$ and remove.
- 3. Repeat until one stream is left (order = Ω)
- Basin is said to be of the order of the last stream removed.
- 5. Example above is a basin of order $\Omega = 3$.

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Stream Ordering—A large example:



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Another way to define ordering:

As before, label all source streams as order ω =
 Follow all labelled streams downstream
 Whenever two streams of the same order (ω) means the resulting stream has order incremented by 1

 If streams of different orders *u*₁ and *u*₂ meet, then the resultant stream has order equal to the largest of the tw



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- Whenever two streams of the same order (ω) meet, the resulting stream has order incremented by 1 (ω + 1).
- If streams of different orders ω₁ and ω₂ meet, then the resultant stream has order equal to the largest of the two.
 Simple rule:

$$\omega_3 = \max(\omega_1, \omega_2) + \delta_{\omega_1, \omega_2}$$

where δ is the Kronecker delta.



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One problem:

Resolution of data messes with ordering

but relationships based on ordering appear to b robusing resolution changes. Branching Networks I

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Utility:

Stream ordering helpfully discretizes a network
 Goal: understand network architecture

Utility:

Stream ordering helpfully discretizes a network.

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Utility:

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Resultant definitions:

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 An order ω basin has a main stream length ℓ_ω.
 An order ω basin has a stream segment length s_ω.
 An order ω stream segment is only that part of the stream which is actually of order ω.
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Three laws:
Horton's law of stream numbers:

 $n_{\omega}/n_{\omega+1}=R_n>1$

Horton's law of stream lengths

 $\left|ar{\ell}_{\omega+1}/ar{\ell}_{\omega}=R_{\ell}>1
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Horton's law of basin areas:

 $ar{a}_{\omega+1}/ar{a}_{\omega}=R_a>1$

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Horton's Ratios:

So... Horton's laws are defined by three ratios:

 R_n, R_ℓ , and R_a .

 $n_{\omega} = n_{\omega-1}/R_0$

 $= n_{\omega-2}/R_n^2$

 $= n_1 / R_n^{\omega - 1}$ $= n_1 e^{-(\omega - 1) \ln R_r}$

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Horton's laws Horton's Ratios:

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$$n_{\omega} = n_{\omega-1}/R_n$$

= $n_{\omega-2}/R_n^2$

.

$$= n_1/R_n^{\omega-1}$$

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Horton's laws Horton's Ratios:

So... Horton's laws are defined by three ratios:

 R_n , R_ℓ , and R_a .

Horton's laws describe exponential decay or growth:

$$n_{\omega} = n_{\omega-1}/R_n$$

= $n_{\omega-2}/R_n^2$

$$= n_1 / R_n^{\omega - 1}$$
$$= n_1 e^{-(\omega - 1) \ln R_n}$$

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Similar story for area and length:

As stream order increases, number drops and are and length increase.

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Similar story for area and length:

$$ar{a}_\omega = ar{a}_1 e^{(\omega-1) \ln R_a}$$

$$\bar{\ell}_{\omega} = \bar{\ell}_1 e^{(\omega-1)\ln R}$$

Ac stream order increases, number drops and area and length increase.

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Similar story for area and length:

$$ar{a}_\omega = ar{a}_1 e^{(\omega-1) \ln R_z}$$

$$\bar{\ell}_{\omega} = \bar{\ell}_1 e^{(\omega-1)\ln R}$$

As stream order increases, number drops and area and length increase.

Branching Networks I





A few more things:

Horton's laws are laws of averages.
 Averaging for number is across basins.
 Averaging for stream lengths and areas is with basins.

 Horton's ratios go a long way to defining a branchin retwork

But we need one other piece of information

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Branching Networks

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A bonus law:

Horton's law of stream segment lengths:

$$ar{s}_{\omega+1}/ar{s}_{\omega}=R_s>1$$

Insert question 2, assignment 2 (\boxplus)

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A bonus law:

Horton's law of stream segment lengths:

 $ar{s}_{\omega+1}/ar{s}_{\omega}=R_{s}>1$

► Can show that R_s = R_ℓ.
 ► Insert question 2, assignment 2 (



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A bonus law:

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Can show that R_s = R_ℓ.
 Insert question 2, assignment 2 (⊞)

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Horton's laws in the real world:



Branching Networks I





Blood networks:

- Horton's laws hold for sections of cardiovascula in networks
- Measuring such networks is tricky and messy.
- Vessel diameters obey an analogous Horton's law

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Blood networks:

 Horton's laws hold for sections of cardiovascular networks Branching Networks I

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Blood networks:

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Blood networks:

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- Measuring such networks is tricky and messy...
- Vessel diameters obey an analogous Horton's law.

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Data from real blood networks

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Network	R _n	R_r^{-1}	R_{ℓ}^{-1}	$-\frac{\ln R_r}{\ln R_r}$	$-\frac{\ln R_{\ell}}{\ln B_{\tau}}$	α	Introduction
West <i>et al.</i>	_	-	_	1/2	1/3	3/4	Definitions Allometry Laws
rat (PAT)	2.76	1.58	1.60	0.45	0.46	0.73	Stream Ordering
cat (PAT) (Turcotte <i>et al.</i> ^[10])	3.67	1.71	1.78	0.41	0.44	0.79	Horton's Laws Tokunaga's Law Nutshell Beferences
dog (PAT)	3.69	1.67	1.52	0.39	0.32	0.90	
pig (LCX) pig (RCA) pig (LAD)	3.57 3.50 3.51	1.89 1.81 1.84	2.20 2.12 2.02	0.50 0.47 0.49	0.62 0.60 0.56	0.62 0.65 0.65	1 F
human (PAT) human (PAT)	3.03 3.36	1.60 1.56	1.49 1.49	0.42 0.37	0.36 0.33	0.83 0.94	University VERMONT

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Observations:

Horton's ratios vary:

R _n	3.0–5.0
Ra	3.0–6.0
R_ℓ	1.5-3.0

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Observations:

Horton's ratios vary:

 $\begin{array}{rrr} R_n & 3.0-5.0 \\ R_a & 3.0-6.0 \\ R_\ell & 1.5-3.0 \end{array}$

No accepted explanation for these values.

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- Horton's laws tell us how quantities vary from level to level ...

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- No accepted explanation for these values.
- Horton's laws tell us how quantities vary from level to level ...
- ... but they don't explain how networks are structured.

Branching Networks

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Nutshell





Delving deeper into network architecture:

 Tokunaga (1968) identified a clearer picture of network structure ^[7, 8, 9]
 As per Horton-Strahler, use stream ordering
 Focus: describe how streams of different order connect to each other.
 Tokunage's law is also a law of averages

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Branching Networks I





Definition:

• $T_{\mu,\nu}$ = the average number of side streams of order ν that enter as tributaries to streams of order μ

 Recall each stream segment of order μ is 'general by two streams of order μ – 1

These generating streams are not considered sic

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Definition:

*T*_{μ,ν} = the average number of side streams of order ν that enter as tributaries to streams of order μ
 μ, ν = 1, 2, 3, ...

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Nutshell





Definition:

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$$\mu, \nu = 1, 2, 3, \dots$$

▶ $\mu \ge \nu + 1$



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Nutshell





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Property 1: Scale independence—depends only on difference between orders:

 Property 2: Number of side streams grows exponentially with difference in orders.

 $T_{\mu,\nu}=T_1(R_T)^{\mu-\nu-}$

Branching Networks I





Property 1: Scale independence—depends only on difference between orders:

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Property 1: Scale independence—depends only on difference between orders:

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Property 1: Scale independence—depends only on difference between orders:

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u} = T_{\mu-
u}$$

Property 2: Number of side streams grows exponentially with difference in orders:

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We usually write Tokunaga's law as:

 $T_k = T_1 (R_T)^{k-1}$ where $R_T \simeq 2$

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Tokunaga's law—an example:

 $T_1 \simeq 2$ $R_T \simeq 4$



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The Mississippi

A Tokunaga graph:



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À.

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- Show remarkable self-similarity over many scales.
- Horton-Strahler Stream ordering gives one usef way of getting at the architecture of branching networks.
- Horton's laws reveal self-similarity.
- Horton's laws can be misinterpreted as suggestin pure hierarchy.
- Tokunaga's laws neatly describe network
- Branching networks exhibit a mixed hierarchica
 - Horton and Tokunaga can be connected analyt

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Branching Networks I

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Branching Networks





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- Horton and Tokunaga can be connected analytically (next up).

Branching Networks





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