Assortativity and Mixing

Complex Networks CSYS/MATH 303, Spring, 2011

Prof. Peter Dodds

Department of Mathematics & Statistics Center for Complex Systems Vermont Advanced Computing Center University of Vermont









VACC



Licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References



20 1 of 26

Outline

Definition

General mixing

Assortativity by degree

Contagion

References

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References



2 0f 2 of 26

- Random networks with arbitrary degree distributions cover much territory but do not represent all networks.
- We can extend in many other directions and a natural one is to introduce correlations between different kinds of nodes.
- Node attributes may be anything, e.g.
 - 2. demographics (age: gender, etc.
 - 3. group atiliation
- We speak of mixing patterns, correlations, biases
 Networks are still random at base but now have n global structure.
- Build on work by Newman^[4, 5], and Boguñá an Sarana ^[1]

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Random networks with arbitrary degree distributions cover much territory but do not represent all networks.
- Moving away from pure random networks was a key first step.
- natural one is to introduce correlations betwee different kinds of nodes.
- Node attributes may be anything, e.g.
 - 2. demographics (age, gender, etc.)
 - We speak of mixing patterns, correlations, biases
 Networks are still random at base but now have n global structure.
- Build on work by Newman^[4, 5], and Boguña and Serang^[1]

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Random networks with arbitrary degree distributions cover much territory but do not represent all networks.
- Moving away from pure random networks was a key first step.
- We can extend in many other directions and a natural one is to introduce correlations between different kinds of nodes.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Random networks with arbitrary degree distributions cover much territory but do not represent all networks.
- Moving away from pure random networks was a key first step.
- We can extend in many other directions and a natural one is to introduce correlations between different kinds of nodes.
- Node attributes may be anything, e.g.:
 - 1. degree
 - 2. demographics (age, gender, etc.)
 - 3. group affiliation

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Random networks with arbitrary degree distributions cover much territory but do not represent all networks.
- Moving away from pure random networks was a key first step.
- We can extend in many other directions and a natural one is to introduce correlations between different kinds of nodes.
- Node attributes may be anything, e.g.:
 - 1. degree
 - 2. demographics (age, gender, etc.)
 - 3. group affiliation
- We speak of mixing patterns, correlations, biases...

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Random networks with arbitrary degree distributions cover much territory but do not represent all networks.
- Moving away from pure random networks was a key first step.
- We can extend in many other directions and a natural one is to introduce correlations between different kinds of nodes.
- Node attributes may be anything, e.g.:
 - 1. degree
 - 2. demographics (age, gender, etc.)
 - 3. group affiliation
- We speak of mixing patterns, correlations, biases...
- Networks are still random at base but now have more global structure.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Random networks with arbitrary degree distributions cover much territory but do not represent all networks.
- Moving away from pure random networks was a key first step.
- We can extend in many other directions and a natural one is to introduce correlations between different kinds of nodes.
- Node attributes may be anything, e.g.:
 - 1. degree
 - 2. demographics (age, gender, etc.)
 - 3. group affiliation
- We speak of mixing patterns, correlations, biases...
- Networks are still random at base but now have more global structure.
- Build on work by Newman^[4, 5], and Boguñá and Serano.^[1].

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



 $e_{\mu
u} = \mathbf{Pr} \left(egin{array}{c} ext{an edge connects a node of type } \mu \ ext{to a node of type }
u \end{array}
ight)$

 $m{a}_{\mu}={\sf Pr}({\sf an} \; {\sf edge} \; {\sf comes} \; {\sf from} \; {\sf a} \; {\sf node} \; {\sf of} \; {\sf type} \; \mu)$

 $b_{
u} = \mathbf{Pr}(an edge leads to a node of type <math>
u$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References



20 A of 26

Assume types of nodes are countable, and are assigned numbers 1, 2, 3,

 $m{e}_{\mu
u}=m{Pr}\left(egin{array}{c} ext{an edge connects a node of type } \mu \ ext{to a node of type }
u \end{array}
ight.$

 $m{a}_{\mu}=m{\mathsf{Pr}}({ ext{an edge comes from a node of type }\mu})$

 $b_{\nu} = \mathbf{Pr}(an edge leads to a node of type <math>\nu$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Assume types of nodes are countable, and are assigned numbers 1, 2, 3,
- Consider networks with directed edges.



Definition

General mixing

Assortativity by degree

Contagion

References

Pr(an edge comes from a node of type

Pr(an edge leads to a node of type a



20 A of 26

- Assume types of nodes are countable, and are assigned numbers 1, 2, 3,
- Consider networks with directed edges.

 $e_{\mu
u} = \mathsf{Pr} \left(egin{array}{c} ext{an edge connects a node of type } \mu \ ext{to a node of type }
u \end{array}
ight)$

r(an edge comes from a node of type

Pr(an edge leads to a node of type

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Assume types of nodes are countable, and are assigned numbers 1, 2, 3,
- Consider networks with directed edges.

 $e_{\mu
u} = \mathsf{Pr} \left(egin{array}{c} ext{an edge connects a node of type } \mu \ ext{to a node of type }
u \end{array}
ight)$

 $a_{\mu} = \mathbf{Pr}($ an edge comes from a node of type μ)



Definition

General mixing

Assortativity by degree

Contagion



- Assume types of nodes are countable, and are assigned numbers 1, 2, 3,
- Consider networks with directed edges.

 $e_{\mu
u} = \mathsf{Pr} \left(egin{array}{c} ext{an edge connects a node of type } \mu \ ext{to a node of type }
u \end{array}
ight)$

 $a_{\mu} = \mathbf{Pr}($ an edge comes from a node of type μ)

 $b_{\nu} = \mathbf{Pr}(an edge leads to a node of type <math>\nu)$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Assume types of nodes are countable, and are assigned numbers 1, 2, 3,
- Consider networks with directed edges.

 $e_{\mu
u} = \mathsf{Pr} \left(egin{array}{c} ext{an edge connects a node of type } \mu \ ext{to a node of type }
u \end{array}
ight)$

 $a_{\mu} = \mathbf{Pr}($ an edge comes from a node of type μ)

 $b_{\nu} = \mathbf{Pr}(an edge leads to a node of type <math>\nu)$

• Write $\mathbf{E} = [e_{\mu\nu}], \vec{a} = [a_{\mu}], \text{ and } \vec{b} = [b_{\nu}].$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Assume types of nodes are countable, and are assigned numbers 1, 2, 3,
- Consider networks with directed edges.

 $e_{\mu
u} = \mathbf{Pr} \left(egin{array}{c} ext{an edge connects a node of type } \mu \ ext{to a node of type }
u \end{array}
ight)$

 $a_{\mu} = \mathbf{Pr}($ an edge comes from a node of type $\mu)$

 $b_{\nu} = \mathbf{Pr}(an edge leads to a node of type <math>\nu)$

• Write $\mathbf{E} = [e_{\mu\nu}], \vec{a} = [a_{\mu}], \text{ and } \vec{b} = [b_{\nu}].$ • Requirements:

$$\sum_{\mu \nu} e_{\mu\nu} = 1, \ \sum_{\nu} e_{\mu\nu} = a_{\mu}, \ \text{and} \sum_{\mu} e_{\mu\nu} = b_{\nu}.$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion





 Perfectly assortative networks where nodes only connect to like-nodes, and the network breaks into subnetworks.

Requires $e_{\mu
u} = 0$ if $\mu \neq \nu$ and $\sum_{\mu} e_{\mu\mu} = 1$.

For these we must have independence: $e_{\mu\nu} = a_{\mu}b_{\nu}$.

 Disassortative networks where nodes connect to modes distinct from themselves

Disassonative networks can be hard to build and may require constraints on the $e_{\mu\nu}$.

 Basic story, level of assortativity reflects the degree to which nodes are connected to nodes within their Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Varying $e_{\mu\nu}$ allows us to move between the following:
 - Perfectly assortative networks where nodes only connect to like nodes, and the network breaks into subnetworks.

2. Uncorrelated networks (as we have studied so far) For these we must have independence: $e_{\mu\nu} = a_{\mu}b_{\nu}$ 3. Disassortative networks where nodes connect to

Disassortative networks can be hard to build and may require constraints on the *e_{inv}*.

University VERMONT

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



• Varying $e_{\mu\nu}$ allows us to move between the following:

1. Perfectly assortative networks where nodes only connect to like nodes, and the network breaks into subnetworks.

Requires $e_{\mu\nu} = 0$ if $\mu \neq \nu$ and $\sum_{\mu} e_{\mu\mu} = 1$.

For these we must have independence: $e_{...} =$

3. Disassortative networks where nodes conne

Disassortative networks can be hard to build and may require constraints on the *e_{inv}*.

 Basic story: level of assortativity reflects the degree to which nodes are connected to nodes within their Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



• Varying $e_{\mu\nu}$ allows us to move between the following:

 Perfectly assortative networks where nodes only connect to like nodes, and the network breaks into subnetworks.

Requires $e_{\mu\nu} = 0$ if $\mu \neq \nu$ and $\sum_{\mu} e_{\mu\mu} = 1$.

- 2. Uncorrelated networks (as we have studied so far)
- 3. Disassortative networks where nodes connect

Usassonative networks can be hard to build an

University Vermont

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



• Varying $e_{\mu\nu}$ allows us to move between the following:

 Perfectly assortative networks where nodes only connect to like nodes, and the network breaks into subnetworks.

Requires $e_{\mu\nu} = 0$ if $\mu \neq \nu$ and $\sum_{\mu} e_{\mu\mu} = 1$.

2. Uncorrelated networks (as we have studied so far) For these we must have independence: $e_{\mu\nu} = a_{\mu}b_{\nu}$. Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



• Varying $e_{\mu\nu}$ allows us to move between the following:

 Perfectly assortative networks where nodes only connect to like nodes, and the network breaks into subnetworks.

Requires $e_{\mu\nu} = 0$ if $\mu \neq \nu$ and $\sum_{\mu} e_{\mu\mu} = 1$.

- 2. Uncorrelated networks (as we have studied so far) For these we must have independence: $e_{\mu\nu} = a_{\mu}b_{\nu}$.
- Disassortative networks where nodes connect to nodes distinct from themselves.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



• Varying $e_{\mu\nu}$ allows us to move between the following:

 Perfectly assortative networks where nodes only connect to like nodes, and the network breaks into subnetworks.

Requires $e_{\mu\nu} = 0$ if $\mu \neq \nu$ and $\sum_{\mu} e_{\mu\mu} = 1$.

- 2. Uncorrelated networks (as we have studied so far) For these we must have independence: $e_{\mu\nu} = a_{\mu}b_{\nu}$.
- Disassortative networks where nodes connect to nodes distinct from themselves.
- Disassortative networks can be hard to build and may require constraints on the $e_{\mu\nu}$.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



• Varying $e_{\mu\nu}$ allows us to move between the following:

 Perfectly assortative networks where nodes only connect to like nodes, and the network breaks into subnetworks.

Requires $e_{\mu\nu} = 0$ if $\mu \neq \nu$ and $\sum_{\mu} e_{\mu\mu} = 1$.

- 2. Uncorrelated networks (as we have studied so far) For these we must have independence: $e_{\mu\nu} = a_{\mu}b_{\nu}$.
- Disassortative networks where nodes connect to nodes distinct from themselves.
- Disassortative networks can be hard to build and may require constraints on the $e_{\mu\nu}$.
- Basic story: level of assortativity reflects the degree to which nodes are connected to nodes within their group.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



 Quantify the level of assortativity with the following assortativity coefficient^[5]:

$$r = \frac{\sum_{\mu} e_{\mu\mu} - \sum_{\mu} a_{\mu} b_{\mu}}{1 - \sum_{\mu} a_{\mu} b_{\mu}} = \frac{\operatorname{Tr} \mathbf{E} - ||\mathbf{E}^2||_1}{1 - ||\mathbf{E}^2||_1}$$

where $|| \cdot ||_1$ is the 1-norm = sum of a matrix's entries.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References



20 C 6 of 26

 Quantify the level of assortativity with the following assortativity coefficient^[5]:

$$r = \frac{\sum_{\mu} e_{\mu\mu} - \sum_{\mu} a_{\mu} b_{\mu}}{1 - \sum_{\mu} a_{\mu} b_{\mu}} = \frac{\operatorname{Tr} \mathbf{E} - ||E^2||_1}{1 - ||E^2||_1}$$

where || · ||₁ is the 1-norm = sum of a matrix's entries.
Tr E is the fraction of edges that are within groups.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



 Quantify the level of assortativity with the following assortativity coefficient^[5]:

$$r = \frac{\sum_{\mu} e_{\mu\mu} - \sum_{\mu} a_{\mu} b_{\mu}}{1 - \sum_{\mu} a_{\mu} b_{\mu}} = \frac{\operatorname{Tr} \mathbf{E} - ||E^2||_1}{1 - ||E^2||_1}$$

where $|| \cdot ||_1$ is the 1-norm = sum of a matrix's entries.

- Tr E is the fraction of edges that are within groups.
- ||E²||₁ is the fraction of edges that would be within groups if connections were random.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



 Quantify the level of assortativity with the following assortativity coefficient^[5]:

$$r = \frac{\sum_{\mu} e_{\mu\mu} - \sum_{\mu} a_{\mu} b_{\mu}}{1 - \sum_{\mu} a_{\mu} b_{\mu}} = \frac{\operatorname{Tr} \mathbf{E} - ||E^2||_1}{1 - ||E^2||_1}$$

where $|| \cdot ||_1$ is the 1-norm = sum of a matrix's entries.

- Tr E is the fraction of edges that are within groups.
- ||E²||₁ is the fraction of edges that would be within groups if connections were random.
- $1 ||E^2||_1$ is a normalization factor so $r_{\text{max}} = 1$.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



 Quantify the level of assortativity with the following assortativity coefficient^[5]:

$$r = \frac{\sum_{\mu} e_{\mu\mu} - \sum_{\mu} a_{\mu} b_{\mu}}{1 - \sum_{\mu} a_{\mu} b_{\mu}} = \frac{\operatorname{Tr} \mathbf{E} - ||E^2||_1}{1 - ||E^2||_1}$$

where $|| \cdot ||_1$ is the 1-norm = sum of a matrix's entries.

- Tr E is the fraction of edges that are within groups.
- ||E²||₁ is the fraction of edges that would be within groups if connections were random.
- $1 ||E^2||_1$ is a normalization factor so $r_{\text{max}} = 1$.
- When Tr $e_{\mu\mu} = 1$, we have r = 1.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



 Quantify the level of assortativity with the following assortativity coefficient^[5]:

$$r = \frac{\sum_{\mu} e_{\mu\mu} - \sum_{\mu} a_{\mu} b_{\mu}}{1 - \sum_{\mu} a_{\mu} b_{\mu}} = \frac{\operatorname{Tr} \mathbf{E} - ||E^2||_1}{1 - ||E^2||_1}$$

where $|| \cdot ||_1$ is the 1-norm = sum of a matrix's entries.

- Tr E is the fraction of edges that are within groups.
- ||E²||₁ is the fraction of edges that would be within groups if connections were random.
- $1 ||E^2||_1$ is a normalization factor so $r_{\text{max}} = 1$.
- When Tr $e_{\mu\mu} = 1$, we have r = 1.
- When $e_{\mu\mu} = a_{\mu}b_{\mu}$, we have r = 0.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



Notes:

r = −1 is inaccessible if three or more types are present.

connected to unlike nodes—no measure of how unlike nodes are

Minimum value of r occurs when all links between

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



Notes:

- r = −1 is inaccessible if three or more types are present.
- Disassortative networks simply have nodes connected to unlike nodes—no measure of how unlike nodes are.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



Notes:

- r = −1 is inaccessible if three or more types are present.
- Disassortative networks simply have nodes connected to unlike nodes—no measure of how unlike nodes are.
- Minimum value of *r* occurs when all links between non-like nodes: $\operatorname{Tr} e_{\mu\mu} = 0$.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



Notes:

- r = −1 is inaccessible if three or more types are present.
- Disassortative networks simply have nodes connected to unlike nodes—no measure of how unlike nodes are.
- Minimum value of *r* occurs when all links between non-like nodes: $\operatorname{Tr} e_{\mu\mu} = 0$.

$$r_{\min} = \frac{-||E^2||_1}{1 - ||E^2||_1}$$

where $-1 \leq r_{\min} < 0$.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



Scalar quantities

 Now consider nodes defined by a scalar integer quantity.

 $e_{ik} = \mathbf{Pr}$ (a randomly chosen edge connects a norwith value *j* to a node with value *k*). *a*_i and *b*_k are defined as before.

 Can now measure correlations between nodes based on this scalar quantity using standard Pearson correlation coefficient (III)

 $=\frac{\sum_{j\,k}j\,k(e_{jk}-a_{j}b_{k})}{\sigma_{a}\sigma_{b}}=\frac{\langle jk\rangle-\langle j\rangle_{a}\langle k\rangle_{b}}{\sqrt{\langle j^{2}\rangle_{a}-\langle j\rangle_{a}^{2}}\sqrt{\langle k^{2}\rangle_{b}-\langle k\rangle_{a}^{2}}}$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References



andomness in the product *jk*.
- Now consider nodes defined by a scalar integer quantity.
- Examples: age in years, height in inches, number of friends, ...
- e_{jk} = Pr (a randomly chosen edge connects a nod with value *j* to a node with value *k*).
 a and b, are defined as before
- Can now measure correlations between nodes based on this scalar quantity using standard Pearson correlation coefficient (III)

$$T = rac{\sum_{j\,k} j\,k(e_{jk} - a_j b_k)}{\sigma_a \sigma_b} = rac{\langle jk \rangle - \langle j \rangle_a \langle k \rangle_b}{\sqrt{\langle j^2 \rangle_a - \langle j \rangle_a^2} \sqrt{\langle k^2 \rangle_b - \langle k \rangle_b^2}}$$

General mixing

Assortativity by degree

Assortativity and

Mixing

Contagion

References



andomness in the product ik

- Now consider nodes defined by a scalar integer quantity.
- Examples: age in years, height in inches, number of friends, ...
- $e_{jk} = \mathbf{Pr}$ (a randomly chosen edge connects a node with value *j* to a node with value *k*).

.Can now measure correlations between nodes based on this scalar quantity using standard Pearson correlation coefficient (⊞):

$$=\frac{\sum_{j\,k}j\,k(e_{jk}-a_{j}b_{k})}{\sigma_{a}\sigma_{b}}=\frac{\langle jk\rangle-\langle j\rangle_{a}\langle k\rangle_{b}}{\sqrt{\langle j^{2}\rangle_{a}-\langle j\rangle_{a}^{2}}\sqrt{\langle k^{2}\rangle_{b}-\langle k\rangle}}$$

VERMONT

ancomness in the product *i*k

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Now consider nodes defined by a scalar integer quantity.
- Examples: age in years, height in inches, number of friends, ...
- $e_{jk} = \mathbf{Pr}$ (a randomly chosen edge connects a node with value *j* to a node with value *k*).
- a_i and b_k are defined as before.

based on this scalar quantity using standard Pearson correlation coefficient (⊞)

$$=\frac{\sum_{j\,k}j\,k(e_{jk}-a_{j}b_{k})}{\sigma_{a}\sigma_{b}}=\frac{\langle jk\rangle-\langle j\rangle_{a}\langle k\rangle_{b}}{\sqrt{\langle j^{2}\rangle_{a}-\langle j\rangle_{a}^{2}}\sqrt{\langle k^{2}\rangle_{b}-\langle k\rangle}}$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Now consider nodes defined by a scalar integer quantity.
- Examples: age in years, height in inches, number of friends, ...
- $e_{jk} = \mathbf{Pr}$ (a randomly chosen edge connects a node with value *j* to a node with value *k*).
- a_i and b_k are defined as before.
- Can now measure correlations between nodes based on this scalar quantity using standard Pearson correlation coefficient (⊞):

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Now consider nodes defined by a scalar integer quantity.
- Examples: age in years, height in inches, number of friends, ...
- $e_{jk} = \mathbf{Pr}$ (a randomly chosen edge connects a node with value *j* to a node with value *k*).
- a_i and b_k are defined as before.
- Can now measure correlations between nodes based on this scalar quantity using standard Pearson correlation coefficient (⊞):

$$r = \frac{\sum_{j\,k} j\,k(e_{jk} - a_j b_k)}{\sigma_a \sigma_b} = \frac{\langle jk \rangle - \langle j \rangle_a \langle k \rangle_b}{\sqrt{\langle j^2 \rangle_a - \langle j \rangle_a^2} \sqrt{\langle k^2 \rangle_b - \langle k \rangle_k^2}}$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



- Now consider nodes defined by a scalar integer quantity.
- Examples: age in years, height in inches, number of friends, ...
- $e_{jk} = \mathbf{Pr}$ (a randomly chosen edge connects a node with value *j* to a node with value *k*).
- a_i and b_k are defined as before.
- Can now measure correlations between nodes based on this scalar quantity using standard Pearson correlation coefficient (⊞):

$$r = \frac{\sum_{j\,k} j\,k(e_{jk} - a_j b_k)}{\sigma_a \sigma_b} = \frac{\langle jk \rangle - \langle j \rangle_a \langle k \rangle_b}{\sqrt{\langle j^2 \rangle_a - \langle j \rangle_a^2} \sqrt{\langle k^2 \rangle_b - \langle k \rangle_b^2}}$$

This is the observed normalized deviation from randomness in the product jk. Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



 Natural correlation is between the degrees of connected nodes.
 Now define e_k with a slight twist;

an edge connects a degree j + 1 node
 lo a degree k + 1 node

 $\begin{pmatrix} an edge runs between a node of in-degree j \\ and a node of out-degree k \end{pmatrix}$

Useful to calculations (as per R_k)
 Important: Must separately define P₀ as the (*e* contain to information about isolated nodes.
 Directed networks still fine but we will assume there on that e_k = e_{k0}.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References



20 P of 26

 Natural correlation is between the degrees of connected nodes.

> an edge connects a degree j + 1 node to a degree k + 1 node

(an edge runs between a node of in-degree *j* and a node of out-degree *k*

Useful to calculations (as per R_k)
 Important: Must separately define R₀ as the (a contain to information about isolated nodes.
 Directed networks still fine but we will assume there on that e_{ik} = e_{ik}.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References



20 P of 26

 Natural correlation is between the degrees of connected nodes.

Now define e_{jk} with a slight twist:

 $e_{jk} = \Pr\left(\begin{array}{c} \text{an edge connects a degree } j+1 \text{ node} \\ \text{to a degree } k+1 \text{ node} \end{array}\right)$

edge runs between a node of in-degree , I a node of out-degree k

Useful for calculations (as per *A_k*)
 Important: Must separately define *P*₀ as the { contain no information about isolated nodes.
 Directed networks still fine but we will assume here on that *e_k = e_k*

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



 Natural correlation is between the degrees of connected nodes.

Now define e_{jk} with a slight twist:

 $e_{jk} = \Pr\left(\begin{array}{c} \text{an edge connects a degree } j+1 \text{ node} \\ \text{to a degree } k+1 \text{ node} \end{array}\right)$

 $= \Pr\left(\begin{array}{c} \text{an edge runs between a node of in-degree } j \\ \text{and a node of out-degree } k \end{array}\right)$

Useful to calculations (as per R_k)
 Important: Must separately define P₀ as the {e_{jk} contain no information about isolated nodes.
 Directed networks still fine but we will assume the here on that e_{jk} = e_{ij}.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



 Natural correlation is between the degrees of connected nodes.

Now define e_{jk} with a slight twist:

 $e_{jk} = \Pr\left(\begin{array}{c} \text{an edge connects a degree } j+1 \text{ node} \\ \text{to a degree } k+1 \text{ node} \end{array}\right)$

 $= \Pr\left(\begin{array}{c} \text{an edge runs between a node of in-degree } j \\ \text{and a node of out-degree } k \end{array}\right)$

Useful for calculations (as per R_k)
 Important international defined of the second seco

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



 Natural correlation is between the degrees of connected nodes.

Now define e_{jk} with a slight twist:

 $e_{jk} = \Pr\left(\begin{array}{c} \text{an edge connects a degree } j+1 \text{ node} \\ \text{to a degree } k+1 \text{ node} \end{array}\right)$

- $= \Pr\left(\begin{array}{c} \text{an edge runs between a node of in-degree } j \\ \text{and a node of out-degree } k \end{array}\right)$
- Useful for calculations (as per R_k)
 Important: Must separately define P₀ as the {e_{jk}} contain no information about isolated nodes.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



 Natural correlation is between the degrees of connected nodes.

Now define e_{jk} with a slight twist:

 $e_{jk} = \Pr\left(\begin{array}{c} \text{an edge connects a degree } j+1 \text{ node} \\ \text{to a degree } k+1 \text{ node} \end{array}\right)$

- $= \Pr\left(\begin{array}{c} \text{an edge runs between a node of in-degree } j \\ \text{and a node of out-degree } k \end{array}\right)$
- Useful for calculations (as per R_k)
- Important: Must separately define P₀ as the {e_{jk}} contain no information about isolated nodes.
- ► Directed networks still fine but we will assume from here on that $e_{jk} = e_{kj}$.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



Notation reconciliation for undirected networks:

$$r = \frac{\sum_{j\,k} j\,k(\boldsymbol{e}_{jk} - \boldsymbol{R}_j \boldsymbol{R}_k)}{\sigma_{\boldsymbol{R}}^2}$$

where, as before, R_k is the probability that a randomly chosen edge leads to a node of degree k + 1, and

$$\sigma_R^2 = \sum_j j^2 R_j - \left[\sum_j j R_j\right]^2$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



Error estimate for *r*:

Remove edge i and recompute r to obtain r_i.

Hepeat for all edges and compute using the jackknite method (\boxplus) [2]

as variables need to be independen ly happy and edges are correlated. Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References



200 11 of 26

Error estimate for r:

- Remove edge i and recompute r to obtain r_i.
- Repeat for all edges and compute using the jackknife method (III)

$$\sigma_r^2 = \sum_i (r_i - r)^2.$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References



200 11 of 26

Error estimate for r:

- Remove edge i and recompute r to obtain r_i.
- Repeat for all edges and compute using the jackknife method (III)

$$\sigma_r^2 = \sum_i (r_i - r)^2.$$

Mildly sneaky as variables need to be independent for us to be truly happy and edges are correlated...

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion



Measurements of degree-degree correlations

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References

VERMONT

	Group	Network	Туре	Size n	Assortativity r	Error σ_r
	a	Physics coauthorship	undirected	52 909	0.363	0.002
Social	a	Biology coauthorship	undirected	1 520 251	0.127	0.0004
	b	Mathematics coauthorship	undirected	253 339	0.120	0.002
	с	Film actor collaborations	undirected	449 913	0.208	0.0002
	d	Company directors	undirected	7 673	0.276	0.004
	е	Student relationships	undirected	573	-0.029	0.037
	f	Email address books	directed	16 881	0.092	0.004
1	g	Power grid	undirected	4 941	-0.003	0.013
Technological	h	Internet	undirected	10 697	-0.189	0.002
	i	World Wide Web	directed	269 504	-0.067	0.0002
	j	Software dependencies	directed	3 162	-0.016	0.020
Electropy	k	Protein interactions	undirected	2 115	-0.156	0.010
Biological	1	Metabolic network	undirected	765	-0.240	0.007
	m	Neural network	directed	307	-0.226	0.016
	n	Marine food web	directed	134	-0.263	0.037
	0	Freshwater food web	directed	92	-0.326	0.031

- Social networks tend to be assortative (homophily)
- Technological and biological networks tend to be disassortative

 Next: Generalize our work for random networks to degree-correlated networks.

activated by one neighbor with probability B_{k1} , we dan handle various problems:

- 1. find the giant component size
- find the probability and extent of spread for simple disease models.
- find the probability of spreading for simple threshold models.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



- Next: Generalize our work for random networks to degree-correlated networks.
- As before, by allowing that a node of degree k is activated by one neighbor with probability B_{k1}, we can handle various problems:

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References



20 13 of 26

- Next: Generalize our work for random networks to degree-correlated networks.
- As before, by allowing that a node of degree k is activated by one neighbor with probability B_{k1}, we can handle various problems:
 - 1. find the giant component size.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



- Next: Generalize our work for random networks to degree-correlated networks.
- As before, by allowing that a node of degree k is activated by one neighbor with probability B_{k1} , we can handle various problems:
 - 1. find the giant component size.
 - 2. find the probability and extent of spread for simple disease models.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



- Next: Generalize our work for random networks to degree-correlated networks.
- As before, by allowing that a node of degree k is activated by one neighbor with probability B_{k1} , we can handle various problems:
 - 1. find the giant component size.
 - 2. find the probability and extent of spread for simple disease models.
 - 3. find the probability of spreading for simple threshold models.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

► Goal: Find $f_{n,j}$ = **Pr** an edge emanating from a degree j + 1 node leads to a finite active subcomponent of size n.

Define $\hat{B}_1 = [B_{k,1}]$. Plan: Find the generating function $F_j(x; \vec{B}_1) = \sum_{n=0}^{\infty} f_{n,j} x^n$.



Da @ 14 of 26

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References

- ► Goal: Find $f_{n,j} = \mathbf{Pr}$ an edge emanating from a degree j + 1 node leads to a finite active subcomponent of size n.
- Repeat: a node of degree k is in the game with probability B_{k1}.

Plane Find the generating function $F_{i}(x; \vec{B}_{1}) = \sum_{n=0}^{\infty} f_{n,i} x^{n}$.



20 14 of 26

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

- ▶ Goal: Find $f_{n,j} = \mathbf{Pr}$ an edge emanating from a degree j + 1 node leads to a finite active subcomponent of size *n*.
- Repeat: a node of degree k is in the game with probability B_{k1}.
- Define $\vec{B}_1 = [B_{k1}]$.



Goal: Find f_{n,j} = Pr an edge emanating from a degree j + 1 node leads to a finite active subcomponent of size n.

- Repeat: a node of degree k is in the game with probability B_{k1}.
- Define $\vec{B}_1 = [B_{k1}]$.

▶ Plan: Find the generating function $F_j(x; \vec{B}_1) = \sum_{n=0}^{\infty} f_{n,j} x^n$.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



Recursive relationship:

$$ar{F}_{j}(x; ec{B}_{1}) = x^{0} \sum_{k=0}^{\infty} rac{m{e}_{jk}}{R_{j}} (1 - B_{k+1,1}) + x \sum_{k=0}^{\infty} rac{m{e}_{jk}}{R_{j}} B_{k+1,1} \left[F_{k}(x; ec{B}_{1})
ight]^{k}$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References

ihe game.
Second term involves Pr we hit an active node which has k outgoing edges.
Next: find average size of active components reached by following a link from a degree / - 1 node



2 C 15 of 26

Recursive relationship:

$$egin{split} F_{j}(x;ec{B}_{1}) &= x^{0}\sum_{k=0}^{\infty}rac{e_{jk}}{R_{j}}(1-B_{k+1,1}) \ &+ x\sum_{k=0}^{\infty}rac{e_{jk}}{R_{j}}B_{k+1,1}\left[F_{k}(x;ec{B}_{1})
ight]^{t} \end{split}$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References

First term = Pr that the first node we reach is not in the game.

VRIVERSITY S

20 15 of 26

Recursive relationship:

$$ar{F}_{j}(x; egin{split} egin{split} egin{aligned} egin{aligned} egin{split} egin{split$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

- First term = Pr that the first node we reach is not in the game.
- Second term involves Pr we hit an active node which has k outgoing edges.



Recursive relationship:

$$ar{F}_{j}(x; egin{split} egin{split} egin{aligned} egin{aligned} egin{split} egin{split$$

- First term = Pr that the first node we reach is not in the game.
- Second term involves Pr we hit an active node which has k outgoing edges.
- Next: find average size of active components reached by following a link from a degree j + 1 node $= F'_j(1; \vec{B}_1)$.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



• Differentiate $F_j(x; \vec{B}_1)$, set x = 1, and rearrange.

component exists. We find:

 $R_j F_j'(1; \vec{B}_1) = \sum_{k=0}^{\infty} e_{jk} B_{k+1,1} + \sum_{k=0}^{\infty} k e_{jk} B_{k+1,1} F_k'(1; \vec{B}_1).$



200 16 of 26

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

• Differentiate $F_j(x; \vec{B}_1)$, set x = 1, and rearrange.

• We use $F_k(1; \vec{B}_1) = 1$ which is true when no giant component exists.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



• Differentiate $F_j(x; \vec{B}_1)$, set x = 1, and rearrange.

• We use $F_k(1; \vec{B}_1) = 1$ which is true when no giant component exists. We find:

$$R_{j}F_{j}'(1;\vec{B}_{1}) = \sum_{k=0}^{\infty} e_{jk}B_{k+1,1} + \sum_{k=0}^{\infty} ke_{jk}B_{k+1,1}F_{k}'(1;\vec{B}_{1}).$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



• Differentiate $F_j(x; \vec{B}_1)$, set x = 1, and rearrange.

• We use $F_k(1; \vec{B}_1) = 1$ which is true when no giant component exists. We find:

$$R_{j}F_{j}'(1;\vec{B}_{1}) = \sum_{k=0}^{\infty} e_{jk}B_{k+1,1} + \sum_{k=0}^{\infty} ke_{jk}B_{k+1,1}F_{k}'(1;\vec{B}_{1}).$$

• Rearranging and introducing a sneaky δ_{jk} :

$$\sum_{k=0}^{\infty} \left(\delta_{jk} R_k - k B_{k+1,1} e_{jk} \right) F'_k(1; \vec{B}_1) = \sum_{k=0}^{\infty} e_{jk} B_{k+1,1}.$$

1 the [O]

20 A 16 of 26

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

In matrix form, we have

$$\mathbf{A}_{\mathbf{E},\vec{B}_1}\vec{F}'(1;\vec{B}_1)=\mathbf{E}\vec{B}_1$$

where

$$\begin{bmatrix} \mathbf{A}_{\mathbf{E},\vec{B}_{1}} \end{bmatrix}_{j+1,k+1} = \delta_{jk}R_{k} - kB_{k+1,1}e_{jk},$$
$$\begin{bmatrix} \vec{F}'(1;\vec{B}_{1}) \end{bmatrix}_{k+1} = F'_{k}(1;\vec{B}_{1}),$$
$$\begin{bmatrix} \mathbf{E} \end{bmatrix}_{j+1,k+1} = e_{jk}, \text{ and } \begin{bmatrix} \vec{B}_{1} \end{bmatrix}_{k+1} = B_{k+1,1}.$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree


So, in principle at least:

$$ec{F}'(1;ec{B}_1) = \mathbf{A}_{\mathbf{E},ec{B}_1}^{-1} \, \mathbf{E} ec{B}_1$$

Now: as $F'(1, B_1)$, the average size of an active component reached along an edge, increases, we move towards a transition to a giant component.

 Right at the transition, the average component size explodes.

Expleding inverses of matrices occur when their determinants are 0.

 $\det A_{\mathbf{E},\hat{\mathbf{R}}} = 0$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References



2 C 18 of 26

So, in principle at least:

$$ec{F}'(1;ec{B}_1) = \mathbf{A}_{\mathbf{E},ec{B}_1}^{-1} \, \mathbf{E} ec{B}_1$$

Now: as $\vec{F}'(1; \vec{B}_1)$, the average size of an active component reached along an edge, increases, we move towards a transition to a giant component.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References



20 18 of 26

So, in principle at least:

$$ec{F}'(1;ec{B}_1) = \mathbf{A}_{\mathbf{E},ec{B}_1}^{-1} \, \mathbf{E} ec{B}_1$$

- Now: as $\vec{F}'(1; \vec{B}_1)$, the average size of an active component reached along an edge, increases, we move towards a transition to a giant component.
- Right at the transition, the average component size explodes.



Assortativity and Mixing

Definition

General mixing

Assortativity by degree



So, in principle at least:

$$ec{F}'(1;ec{B}_1) = \mathbf{A}_{\mathbf{E},ec{B}_1}^{-1} \, \mathbf{E} ec{B}_1$$

- Now: as $\vec{F}'(1; \vec{B}_1)$, the average size of an active component reached along an edge, increases, we move towards a transition to a giant component.
- Right at the transition, the average component size explodes.
- Exploding inverses of matrices occur when their determinants are 0.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



So, in principle at least:

$$ec{F}'(1;ec{B}_1) = \mathbf{A}_{\mathbf{E},ec{B}_1}^{-1} \, \mathbf{E} ec{B}_1.$$

- Now: as $\vec{F}'(1; \vec{B}_1)$, the average size of an active component reached along an edge, increases, we move towards a transition to a giant component.
- Right at the transition, the average component size explodes.
- Exploding inverses of matrices occur when their determinants are 0.
- The condition is therefore:

$$\det \mathbf{A}_{\mathbf{E},\vec{B}_1} = 0$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



General condition details:

$$\det \mathbf{A}_{\mathbf{E},\vec{B}_{1}} = \det \left[\delta_{jk} R_{k-1} - (k-1) B_{k,1} e_{j-1,k-1} \right] = 0.$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References

When $\vec{B}_1 = B\vec{1}$, we have the condition for a simple disease model's successful spread

 $let[\delta_{jk}R_{k-1} - B(k-1)e_{j-1,k-1}] =$

When $\vec{B}_{1} = \vec{1}$, we have the condition for the existence of a giant component:



200 19 of 26

General condition details:

$$\det \mathbf{A}_{\mathbf{E},\vec{B}_{1}} = \det \left[\delta_{jk} R_{k-1} - (k-1) B_{k,1} e_{j-1,k-1} \right] = 0.$$

• The above collapses to our standard contagion condition when $e_{jk} = R_j R_k$.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References



200 19 of 26

General condition details:

$$\det \mathbf{A}_{\mathbf{E},\vec{B}_{1}} = \det \left[\delta_{jk} R_{k-1} - (k-1) B_{k,1} e_{j-1,k-1} \right] = 0.$$

• The above collapses to our standard contagion condition when $e_{jk} = R_j R_k$.

• When $\vec{B}_1 = B\vec{1}$, we have the condition for a simple disease model's successful spread

$$\det\left[\delta_{jk}R_{k-1}-B(k-1)e_{j-1,k-1}\right]=0.$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



General condition details:

$$\det \mathbf{A}_{\mathbf{E},\vec{B}_{1}} = \det \left[\delta_{jk} R_{k-1} - (k-1) B_{k,1} e_{j-1,k-1} \right] = 0.$$

• The above collapses to our standard contagion condition when $e_{jk} = R_j R_k$.

• When $\vec{B}_1 = B\vec{1}$, we have the condition for a simple disease model's successful spread

$$\det\left[\delta_{jk}R_{k-1}-B(k-1)e_{j-1,k-1}\right]=0.$$

When $\vec{B}_1 = \vec{1}$, we have the condition for the existence of a giant component:

$$\det\left[\delta_{jk}R_{k-1}-(k-1)e_{j-1,k-1}\right]=0.$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



General condition details:

$$\det \mathbf{A}_{\mathbf{E},\vec{B}_{1}} = \det \left[\delta_{jk} R_{k-1} - (k-1) B_{k,1} e_{j-1,k-1} \right] = 0.$$

• The above collapses to our standard contagion condition when $e_{jk} = R_j R_k$.

• When $\vec{B}_1 = B\vec{1}$, we have the condition for a simple disease model's successful spread

$$\det\left[\delta_{jk}R_{k-1}-B(k-1)e_{j-1,k-1}\right]=0.$$

When $\vec{B}_1 = \vec{1}$, we have the condition for the existence of a giant component:

$$\det \left[\delta_{jk} R_{k-1} - (k-1) e_{j-1,k-1} \right] = 0.$$

Bonusville: We'll find a much better version of this set of conditions later...

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



We'll next find two more pieces:

1. P_{trig}, the probability of starting a cascade

Triggering probability:

Generating function

 Generating function for vulnerable component size is more complicated. Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References



We'll next find two more pieces:

- 1. Ptrig, the probability of starting a cascade
- 2. *S*, the expected extent of activation given a small seed.

Triggering probability:

Generating function

 Generating function for vulnerable component size is more complicated.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References



We'll next find two more pieces:

- 1. P_{trig}, the probability of starting a cascade
- 2. *S*, the expected extent of activation given a small seed.

Triggering probability:

Generating function:

$$H(x; \vec{B}_1) = x \sum_{k=0}^{\infty} P_k \left[F_{k-1}(x; \vec{B}_1) \right]^k.$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References



We'll next find two more pieces:

- 1. P_{trig}, the probability of starting a cascade
- 2. *S*, the expected extent of activation given a small seed.

Triggering probability:

Generating function:

$$H(x; \vec{B}_1) = x \sum_{k=0}^{\infty} P_k \left[F_{k-1}(x; \vec{B}_1) \right]^k.$$

 Generating function for vulnerable component size is more complicated.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References



Want probability of not reaching a finite component.

$$P_{\text{trig}} = S_{\text{trig}} = 1 - H(1; \vec{B}_1)$$

= 1 - $\sum_{k=0}^{\infty} P_k \left[F_{k-1}(1; \vec{B}_1) \right]^k$.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References

VERMONT

2 0 0 21 of 26

Want probability of not reaching a finite component.

$$P_{\text{trig}} = S_{\text{trig}} = 1 - H(1; \vec{B}_1)$$

= 1 - $\sum_{k=0}^{\infty} P_k \left[F_{k-1}(1; \vec{B}_1) \right]^k$.

• Last piece: we have to compute $F_{k-1}(1; \vec{B}_1)$.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



Want probability of not reaching a finite component.

$$P_{\text{trig}} = S_{\text{trig}} = 1 - H(1; \vec{B}_1)$$

= 1 - $\sum_{k=0}^{\infty} P_k \left[F_{k-1}(1; \vec{B}_1) \right]^k$.

• Last piece: we have to compute $F_{k-1}(1; \vec{B}_1)$.

Nastier (nonlinear)—we have to solve the recursive expression we started with when x = 1: $F_j(1; \vec{B}_1) = \sum_{k=0}^{\infty} \frac{e_{jk}}{R_j} (1 - B_{k+1,1}) +$

$$\sum_{k=0}^{\infty} \frac{e_{jk}}{R_j} B_{k+1,1} \left[F_k(1; \vec{B}_1) \right]^k$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



Want probability of not reaching a finite component.

$$P_{\text{trig}} = S_{\text{trig}} = 1 - H(1; \vec{B}_1)$$

= 1 - $\sum_{k=0}^{\infty} P_k \left[F_{k-1}(1; \vec{B}_1) \right]^k$.

• Last piece: we have to compute $F_{k-1}(1; \vec{B}_1)$.

Nastier (nonlinear)—we have to solve the recursive expression we started with when x = 1: $F_j(1; \vec{B}_1) = \sum_{k=0}^{\infty} \frac{e_{jk}}{B_j} (1 - B_{k+1,1}) +$

$$\sum_{k=0}^{\infty} \frac{e_{jk}}{B_j} B_{k+1,1} \left[F_k(1; \vec{B}_1) \right]'$$

Iterative methods should work here.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



Truly final piece: Find final size using approach of Gleeson^[3], a generalization of that used for uncorrelated random networks.

leading to a degree / node is infected at time t.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References



- Truly final piece: Find final size using approach of Gleeson^[3], a generalization of that used for uncorrelated random networks.
- Need to compute θ_{j,t}, the probability that an edge leading to a degree j node is infected at time t.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



- Truly final piece: Find final size using approach of Gleeson^[3], a generalization of that used for uncorrelated random networks.
- Need to compute θ_{j,t}, the probability that an edge leading to a degree j node is infected at time t.

Evolution of edge activity probability:

$$heta_{j,t+1} = G_{j}(ec{ heta_{t}}) = \phi_{0} + (1-\phi_{0}) imes$$

$$\sum_{k=1}^{\infty} \frac{e_{j-1,k-1}}{R_{j-1}} \sum_{i=0}^{k-1} {\binom{k-1}{i}} \theta_{k,t}^i (1-\theta_{k,t})^{k-1-i} B_{ki}.$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



- Truly final piece: Find final size using approach of Gleeson^[3], a generalization of that used for uncorrelated random networks.
- Need to compute θ_{j,t}, the probability that an edge leading to a degree j node is infected at time t.

Evolution of edge activity probability:

$$heta_{j,t+1} = G_{j}(ec{ heta}_{t}) = \phi_0 + (1-\phi_0) imes$$

$$\sum_{k=1}^{\infty} \frac{e_{j-1,k-1}}{R_{j-1}} \sum_{i=0}^{k-1} \binom{k-1}{i} \theta_{k,t}^{i} (1-\theta_{k,t})^{k-1-i} B_{ki}.$$

Overall active fraction's evolution:

$$\phi_{t+1} = \phi_0 + (1 - \phi_0) \sum_{k=0}^{\infty} P_k \sum_{i=0}^{k} \binom{k}{i} \theta_{k,t}^i (1 - \theta_{k,t})^{k-i} B_{ki}$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References



As before, these equations give the actual evolution of φ_t for synchronous updates. Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References



If $G_1(0) = 0$ v, want largest eigenvalue $\begin{bmatrix} \frac{\partial G_1(0)}{\partial \theta_{k,t}} \end{bmatrix}$

elgenvalues of this matrix:

Insert question from assignment 9 (\boxplus)



- As before, these equations give the actual evolution of φ_t for synchronous updates.
- Contagion condition follows from $\vec{\theta}_{t+1} = \vec{G}(\vec{\theta}_t)$.



Definition

General mixing

Assortativity by degree

Contagion References

Insert question from assignment 9 (\boxplus)



- As before, these equations give the actual evolution of φ_t for synchronous updates.
- Contagion condition follows from $\vec{\theta}_{t+1} = \vec{G}(\vec{\theta}_t)$.

• Expand \vec{G} around $\vec{\theta}_0 = \vec{0}$.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References

Insert question from assignment 9 (\boxplus)



- As before, these equations give the actual evolution of φ_t for synchronous updates.
- Contagion condition follows from $\vec{\theta}_{t+1} = \vec{G}(\vec{\theta}_t)$.
- Expand \vec{G} around $\vec{\theta}_0 = \vec{0}$.

$$\theta_{j,t+1} = G_j(\vec{0}) + \sum_{k=1}^{\infty} \frac{\partial G_j(\vec{0})}{\partial \theta_{k,t}} \theta_{k,t} + \frac{1}{2!} \sum_{k=1}^{\infty} \frac{\partial^2 G_j(\vec{0})}{\partial \theta_{k,t}^2} \theta_{k,t}^2 + \dots$$

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion References

Insert question from assignment 9 (\boxplus)



- As before, these equations give the actual evolution of \(\phi_t\) for synchronous updates.
- Contagion condition follows from $\vec{\theta}_{t+1} = \vec{G}(\vec{\theta}_t)$.
- Expand \vec{G} around $\vec{\theta}_0 = \vec{0}$.

$$\theta_{j,t+1} = G_j(\vec{0}) + \sum_{k=1}^{\infty} \frac{\partial G_j(\vec{0})}{\partial \theta_{k,t}} \theta_{k,t} + \frac{1}{2!} \sum_{k=1}^{\infty} \frac{\partial^2 G_j(\vec{0})}{\partial \theta_{k,t}^2} \theta_{k,t}^2 + \dots$$

- If G_j(0) ≠ 0 for at least one j, always have some infection.
- ► If $G_j(\vec{0}) = 0 \forall j$, want largest eigenvalue $\left| \frac{\partial G_j(\vec{0})}{\partial \theta_{k,t}} \right| > 1$.

Insert question from assignment 9 (\boxplus)

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



- As before, these equations give the actual evolution of \(\phi_t\) for synchronous updates.
- Contagion condition follows from $\vec{\theta}_{t+1} = \vec{G}(\vec{\theta}_t)$.
- Expand \vec{G} around $\vec{\theta}_0 = \vec{0}$.

$$\theta_{j,t+1} = G_j(\vec{0}) + \sum_{k=1}^{\infty} \frac{\partial G_j(\vec{0})}{\partial \theta_{k,t}} \theta_{k,t} + \frac{1}{2!} \sum_{k=1}^{\infty} \frac{\partial^2 G_j(\vec{0})}{\partial \theta_{k,t}^2} \theta_{k,t}^2 + \dots$$

- If G_j(0) ≠ 0 for at least one j, always have some infection.
- ► If $G_j(\vec{0}) = 0 \forall j$, want largest eigenvalue $\left| \frac{\partial G_j(\vec{0})}{\partial \theta_{k,t}} \right| > 1$.
- Condition for spreading is therefore dependent on eigenvalues of this matrix:

$$\frac{\partial G_j(\vec{0})}{\partial \theta_{k,t}} = \frac{e_{j-1,k-1}}{R_{j-1}}(k-1)B_{k1}$$

Insert question from assignment 9 (⊞)

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



How the giant component changes with assortativity:



from Newman, 2002^[4]

More assortative networks percolate for lower average degrees

 But disassortative networks end up with higher extents of spreading.

Assortativity and Mixing

Definition

General mixing

Assortativity by degree



References I

[1] M. Boguñá and M. Ángeles Serrano.
Generalized percolation in random directed networks.
Phys. Rev. E, 72:016106, 2005. pdf (⊞)

[2] B. Efron and C. Stein.
The jackknife estimate of variance.
The Annals of Statistics, 9:586–596, 1981. pdf (⊞)

[3] J. P. Gleeson.

Cascades on correlated and modular random networks. Phys. Rev. E. 77:046117, 2008. pdf (⊞)

[4] M. Newman.
Assortative mixing in networks.
Phys. Rev. Lett., 89:208701, 2002. pdf (⊞)

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References



References II

Assortativity and Mixing

Definition

General mixing

Assortativity by degree

Contagion

References

[5] M. E. J. Newman. Mixing patterns in networks. Phys. Rev. E, 67:026126, 2003. pdf (⊞)

