Mechanisms for Generating Power-Law Distributions Principles of Complex Systems CSYS/MATH 300, Fall, 2010

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Power-Law Mechanisms

Random Walks The First Return Problem Examples

Variable transformation Basics Holtsmark's Distribution PLIPLO

Growth Mechanisms Random Copying Words, Cities, and the Web





Outline

Random Walks The First Return Problem Examples Variable transformation Basics Holtsmark's Distribution PLIPLO Growth Mechanisms Random Copying Words, Cities, and the Web

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Mechanisms

A powerful story in the rise of complexity:

structure arises out of randomness.

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Mechanisms

A powerful story in the rise of complexity:

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- ► Exhibit A: Random walks... (⊞)

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The essential random walk:

One spatial dimension.

- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin x = 0.
- Step at time t is ϵ_t :

1 with probability 1/2 1 with probability 1/2

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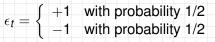
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Displacement after t steps:

$$\mathbf{x}_t = \sum_{i=1}^r \epsilon_i$$

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Expected displacement:



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Displacement after *t* steps:

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$$\langle \mathbf{x}_t \rangle = \left\langle \sum_{i=1}^r \epsilon_i \right\rangle = \sum_{i=1}^r \langle \epsilon_i \rangle = 0$$

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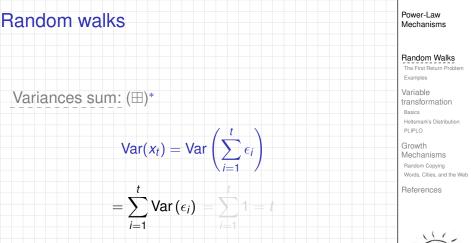
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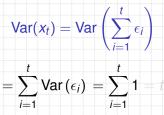


* Sum rule = a good reason for using the variance to measure spread; only works for independent distributions.





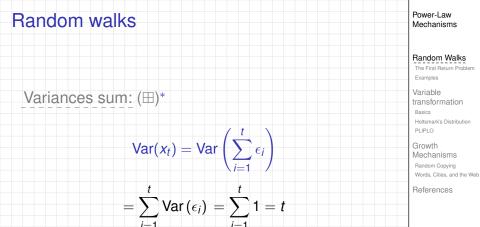
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So typical displacement from the origin scales as

 $\sigma = t^{1/2}$

A non-trivial power-law arises out of additive aggregation or accumulation Mechanisms Random Walks The First Return Problem Examples Variable transformation Basics

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Random walks are weirder than you might think ...

For example:

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See Feller, ^[3] Intro to Probability Theory, Volume I

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Random Walks

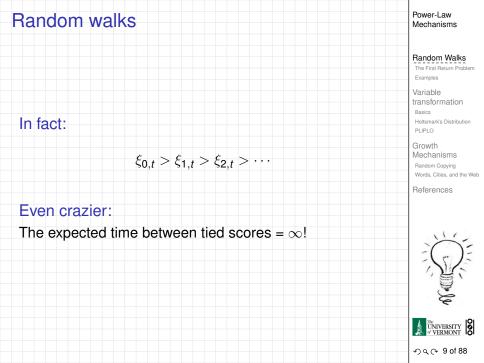
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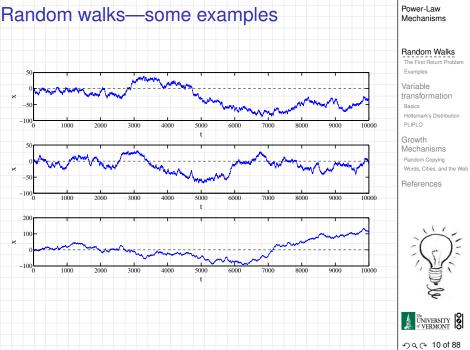
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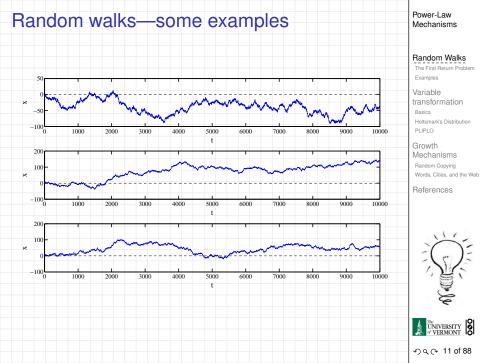
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Random walks	Power-Law Mechanisms
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In fact:	Variable transformation Basics Holtsmark's Distribution PLIPLO
$\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$	Growth Mechanisms Random Copying Words, Cities, and the Web
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The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?

What about higher dimensions?

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First returns

Reasons for caring:

- We will find a power-law size distribution with an interesting exponent
- Some physical structures may result from randor walks
- We'll start to see how different scalings relate to each other

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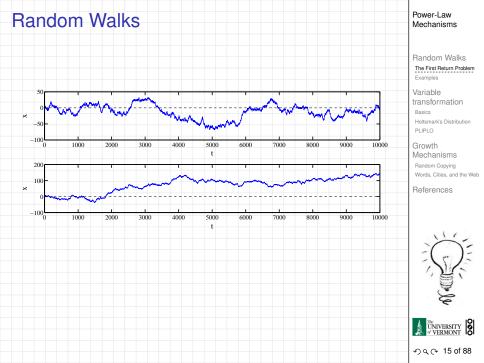


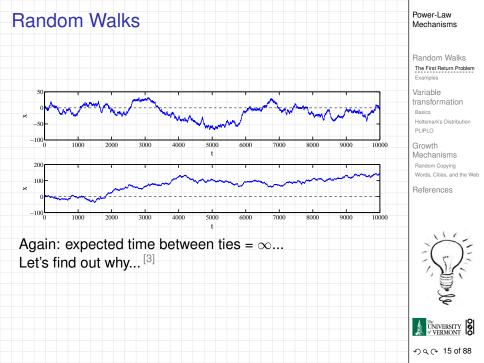
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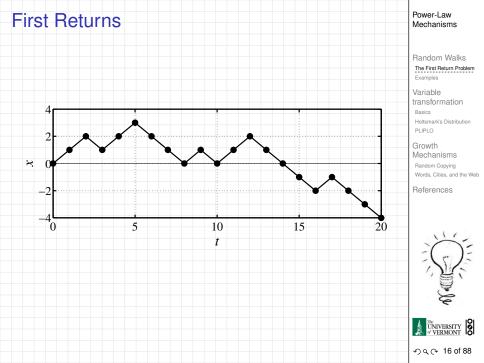
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Power-Law Mechanisms Random Walks The First Return Problem Examples For random walks in 1-d: Variable transformation Basics • Return can only happen when t = 2n. Holtsmark's Distribution PLIPLO Growth Mechanisms Random Copying Words, Cities, and the Web References UNIVERSITY A A 17 of 88

For random walks in 1-d:

- Return can only happen when t = 2n.
- Call $P_{\text{first return}}(2n) = P_{\text{fr}}(2n)$ probability of first return at t = 2n.

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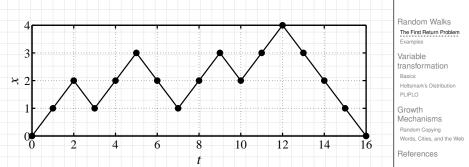
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- Assume drunkard first lurches to x = 1.
- The problem

$$P_{\rm fr}(2n) = 2Pr(x_t \ge 1, t = 1, \dots, 2n - 1, \text{ and } x_{2n} = 0)$$

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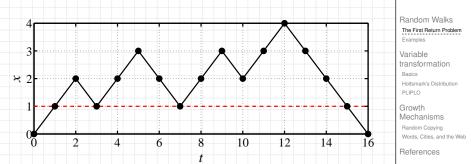
- A useful restatement: $P_{\rm fr}(2n) =$
 - $2 \cdot \frac{1}{2} Pr(x_t \ge 1, t = 1, \dots, 2n 1, \text{ and } x_1 = x_{2n-1} = 1$
- Want walks that can return many times to x = 1.
- (The $\frac{1}{2}$ accounts for stepping to 2 instead of 0 at t = 2n.)



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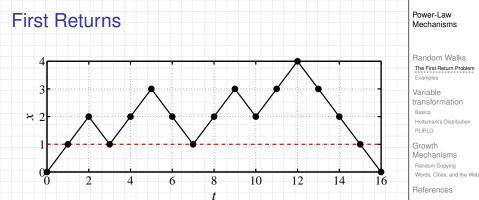
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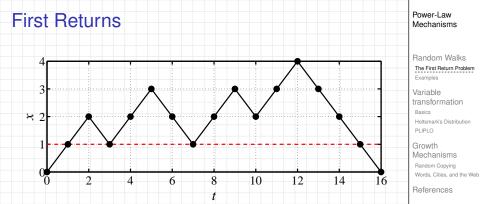
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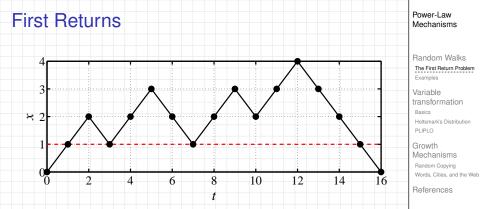
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Counting problem (combinatorics/statistical mechanics)

Use a method of images

- Define N(i, j, t) as the # of possible walks between x = i and x = i taking t steps
- Consider all paths starting at x = 1 and ending at x = 1 after t = 2n 2 steps.
- Subtract how many hit x = 0.

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- Counting problem (combinatorics/statistical mechanics)
- Use a method of images
- Define N(i, j, t) as the # of possible walks between x = i and x = j taking t steps.
- Consider all paths starting at x = 1 and ending at x = 1 after t = 2n - 2 steps.
- Subtract how many hit x = 0.

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Power-Law First Returns Mechanisms Random Walks The First Return Problem Examples Variable transformation Key observation: Basics # of *t*-step paths starting and ending at x = 1Holtsmark's Distribution PLIPLO and hitting x = 0 at least once Growth Mechanisms Random Copying Words, Cities, and the Web References UNIVERSITY ∽ < (~ 20 of 88

Key observation:

of *t*-step paths starting and ending at x = 1

and hitting x = 0 at least once

= # of *t*-step paths starting at x = -1 and ending at x = 1

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Key observation:

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Key observation:

of *t*-step paths starting and ending at x = 1and hitting x = 0 at least once = # of *t*-step paths starting at x = -1 and ending at x = 1= N(-1, 1, t)

So $N_{\text{first return}}(2n) = N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)$

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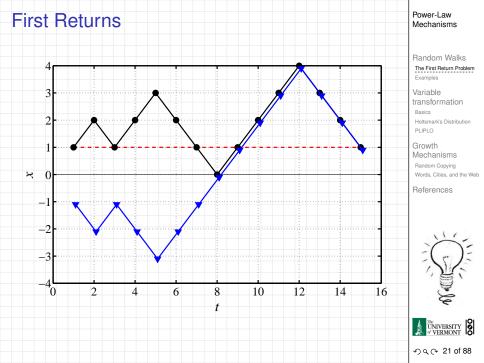
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See this 1-1 correspondence visually...

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For any path starting at x = 1 that hits 0, there is a unique matching path starting at x = -1.

Matching path first mirrors and then tracks.

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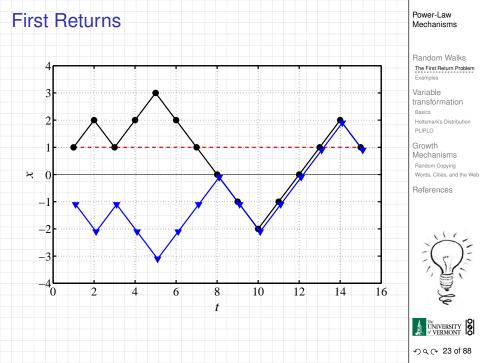
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• Next problem: what is N(i, j, t)?

- # positive steps + # negative steps =
- Random walk must displace by j i after t steps.
- # positive steps # negative steps = j i.
- # positive steps = (t + j i)/2.



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- # positive steps = (t + j i)/2.

$$N(i, j, t) = \begin{pmatrix} t \\ \# \text{ positive steps} \end{pmatrix} = \begin{pmatrix} t \\ (t+j-i)/2 \end{pmatrix}$$

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We now have

$$N_{\text{first return}}(2n) = N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)$$

where

$$N(i,j,t) = \binom{t}{(t+j-i)/2}$$

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Insert question from assignment 4 (\boxplus) Find $N_{\text{first return}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$.

Normalized Number of Paths gives Probability

Total number of possible paths = 2²



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$$P_{\text{first return}}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$

$$I_{n}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$
$$1 = 2^{2n-3/2}$$

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$$\simeq rac{1}{2^{2n}} rac{2^{2n-3/2}}{\sqrt{2\pi} n^{3/2}}$$

$$=\frac{1}{\sqrt{2\pi}}(2n)^{-3/2}$$

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Same scaling holds for continuous space/time walks.

- P(t) is normalizable
- Recurrence: Random walker always returns to origin
- Moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

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Higher dimensions:

Walker in d = 2 dimensions must also return

- Walker may not return in $d \ge 3$ dimensions
- For d = 1, $\gamma = 3/2 \rightarrow \langle t \rangle = \infty$
- Even though walker must return, expect a long wait...

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Words, Cities, and the Web



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$$d=1,\,\gamma=3/2
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Even though walker must return, expect a long wait...

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On finite spaces:

- In any finite volume, a random walker will visit every site with equal probability
- Random walking Diffusion
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

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Random walks on

frequency \propto node degree

Equal probability still present:

On networks, a random walker visits each node with

On networks:

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Random walks on

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On networks:

- \blacktriangleright On networks, a random walker visits each node with frequency \propto node degree
- Equal probability still present: walkers traverse edges with equal frequency.

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Scheidegger Networks [11, 2]



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Triangular lattice

 'Flow' is southeast or southwest with equal probability.

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Scheidegger Networks

Creates basins with random walk boundaries

- Observe Subtracting one random walk from an gives random walk with increments
- Basin length ℓ distribution: $\mathcal{P}(\ell) \propto \ell^{-3/2}$

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Scheidegger Networks

- Creates basins with random walk boundaries
- Observe Subtracting one random walk from another gives random walk with increments

 $\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$

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Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$

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Connections between Exponents For a basin of length ℓ , width $\propto \ell^{1/2}$ Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$ Invert: $\ell \propto a^{2/3}$ $d\ell \propto d(a^{2/3}) = 2/3a^{-1/3}da$ Pr(basin area = a)da

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- Invert: $\ell \propto a^{2/3}$
- $d\ell \propto d(a^{2/3}) = 2/3a^{-1/3}da$
- Pr(basin area = a)da
 - $= Pr(basin length = \ell)d$
 - o(1776) $-(a^{2/3})^{-3/2}a^{-1/3}$
 - $= a^{-4/3} da$
 - soft a state

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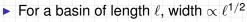
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For a basin of length ℓ , width $\propto \ell^{1/2}$

▶ Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

• $d\ell \propto d(a^{2/3}) = 2/3a^{-1/3}da$

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$ngth = \ell d\ell$	

linvert: $\ell \propto a^{2/3}$

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For a	basin	of	length	l,	width	$\propto \ell$	1/2
			Ų				

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- Pr(basin area = a)da
 - $= Pr(\text{basin length} = \ell)d\ell$
 - $\propto l^{-3/2} dl$ $\propto (a^{2/3})^{-3/2} a^{-1/2}$
 - $= a^{-4/3} da$
 - $= a^{-\tau} da$

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For a basin of length ℓ , width $\propto \ell^{1/2}$ ▶ Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

- linvert: $\ell \propto a^{2/3}$
- $d\ell \propto d(a^{2/3}) = 2/3a^{-1/3}da$
- \blacktriangleright *Pr*(basin area = *a*)d*a*
 - $= Pr(basin length = \ell)d\ell$
 - $\propto \ell^{-3/2} d\ell$

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For a basin	of lenath	ℓ . width	$\propto \ell^{1/2}$
	3	.,	

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$$d\ell \propto d(a^{2/3}) = 2/3a^{-1/3}da$$

• Pr(basin area = a)da

 $= \Pr(\text{basin length} = \ell) d\ell$

$$\propto \ell^{-3/2} d\ell \propto (a^{2/3})^{-3/2} a^{-1/3} da$$

 $= a^{-4/3} da$



For a basin of length ℓ , width $\propto \ell^{1/2}$

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▶ Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$

Pr(basin area = a)da $= Pr(basin length = \ell)d\ell$ $\propto \ell^{-3/2} d\ell$ $\propto (a^{2/3})^{-3/2}a^{-1/3}da$ $= a^{-4/3} da$

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For a	basin d	of leng	jth ℓ, v	vidth	$\propto \ell$	1/2
	area a					

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 $= \Pr(\text{basin length} = \ell) d\ell$

$$\propto \ell^{-3/2} d\ell$$

 $\propto (a^{2/3})^{-3/2} a^{-1/3} da$

- $= a^{-4/3} da$
- $= a^{-\tau} da$

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Both basin area and length obey power law distributions

- Observed for real river networks
- Typically: 1.3 < au < 1.5 and 1.5 < au < 2
- Smaller basins more allometric (h > 1/2)
- Larger basins more isometric (h = 1/2)

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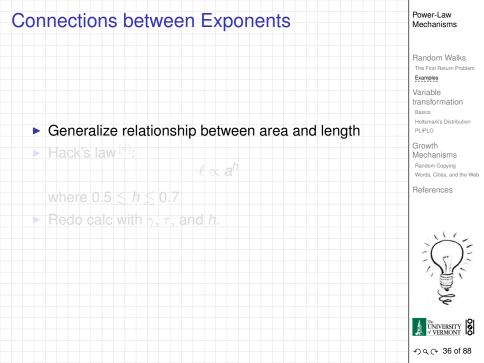
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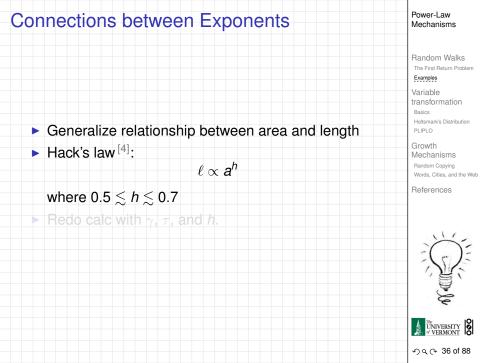
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Connections between Exponents Generalize relationship between area and length Hack's law^[4]: $\ell \propto a^h$ where $0.5 \leq h \leq 0.7$ Redo calc with γ , τ , and h.

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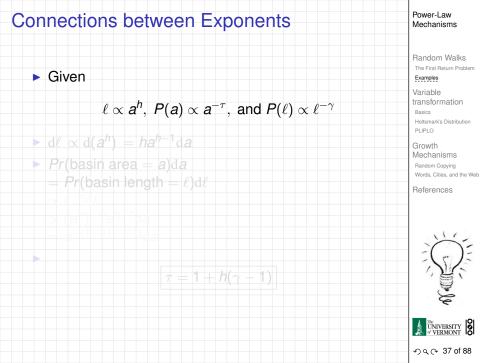
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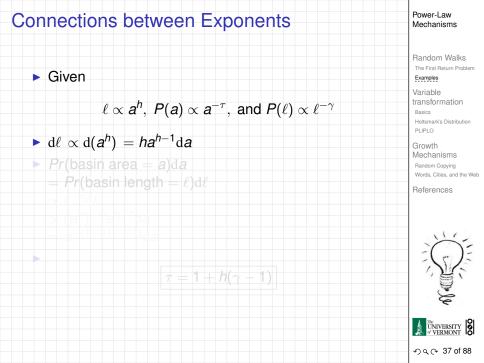
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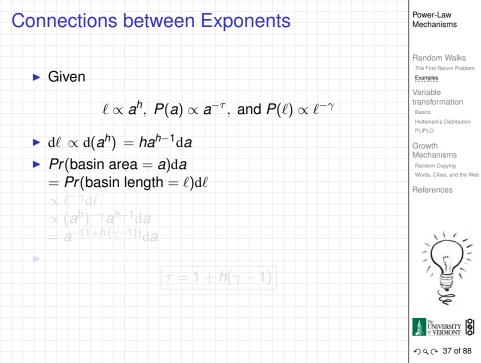
References

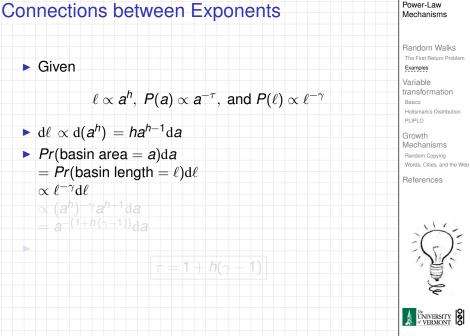


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Power-Law Connections between Exponents Mechanisms Random Walks The First Return Problem Given Examples Variable transformation $\ell \propto a^h$, $P(a) \propto a^{-\tau}$, and $P(\ell) \propto \ell^{-\gamma}$ Rasics Holtsmark's Distribution PLIPLO $\blacktriangleright d\ell \propto d(a^h) = ha^{h-1} da$ Growth Mechanisms Pr(basin area = a)da Random Copying Words, Cities, and the Web $= Pr(basin length = \ell)d\ell$ References $\propto \ell^{-\gamma} \mathrm{d} \ell$ $\propto (a^h)^{-\gamma} a^{h-1} \mathrm{d}a$ UNIVERSITY √ < ?? of 88</p>

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Power-Law Connections between Exponents Mechanisms Random Walks The First Return Problem Given Examples Variable transformation $\ell \propto a^h$, $P(a) \propto a^{-\tau}$, and $P(\ell) \propto \ell^{-\gamma}$ Rasics Holtsmark's Distribution PLIPLO $\blacktriangleright d\ell \propto d(a^h) = ha^{h-1} da$ Growth Mechanisms Pr(basin area = a)da Random Copying Words, Cities, and the Web $= Pr(basin length = \ell)d\ell$ References $\propto \ell^{-\gamma} \mathrm{d} \ell$ $\propto (a^h)^{-\gamma} a^{h-1} \mathrm{d}a$ $= a^{-(1+h(\gamma-1))} da$ Þ $\overline{\tau} = \mathbf{1} + h(\gamma - \mathbf{1})$

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With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies:

$$\tau = 2 - I$$

$$\gamma = 1/h$$

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$$au = \mathbf{2} - \mathbf{h}$$

$$\gamma = 1/h$$

Only one exponent is independent

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- Only one exponent is independent
- Simplify system description

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- Expect scaling relations where power laws are found

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$$\gamma = 1/h$$

- Only one exponent is independent
- Simplify system description
- Expect scaling relations where power laws are found
- Characterize universality class with independent exponents

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Other First Returns

Failure

- A very simple model of failure/death:
- x_t = entity's 'health' at time t
- x₀ could be > 0.
- Entity fails when x hits 0.

Streams

- Dispersion of suspended sediments in streams.
- Long times for clearing.

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Other First Returns

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In 1-d.

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Can generalize to Fractional Random Walks

Levy flights, Fractional Brownian Motion

Extensive memory of path now matters...

▶ In 1-d.

Can generalize to Fractional Random Walks

Levy flights, Fractional Brownian Motion

Extensive memory of path now matters...

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Can generalize to Fractional Random Walks

- Levy flights, Fractional Brownian Motion
- In 1-d,

 $\sigma \sim t^{\alpha}$

Extensive memory of path now matters...

Power-Law Mechanisms Random Walks

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Variable transformation Basics Holtsmark's Distribution

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Understand power laws as arising from

1. elementary distributions (e.g., exponentials)

Understand power laws as arising from

1. elementary distributions (e.g., exponentials)

2. variables connected by power relationships

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Random variable X with known distribution P_x

- Second random variable Y with y = f(x).
- $\triangleright P_y(y) dy = P_x(x) dx$
 - $\sum_{y \mid f(x) = y} P_{x}(f^{-1}(y)) \xrightarrow{dy} Figure...$
- Often easier to do by hand...

Power-Law Mechanisms

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Variable transformation

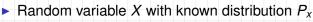
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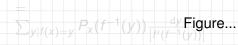
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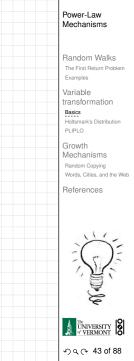


Second random variable Y with y = f(x).



 Often easier to do by hand...

 $P_{y}(y)dy = P_{x}(x)dx$



Random variable X with known distribution P_x

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√ < ? ~ 43 of 88</p>

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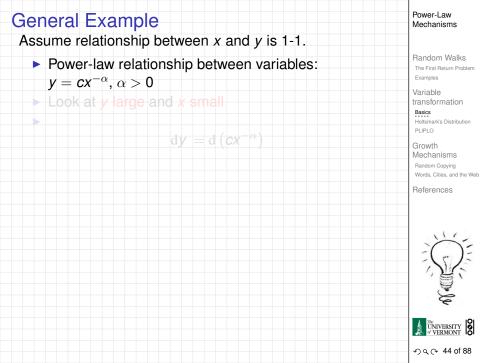
$$P_y(y)dy = P_x(x)dx$$

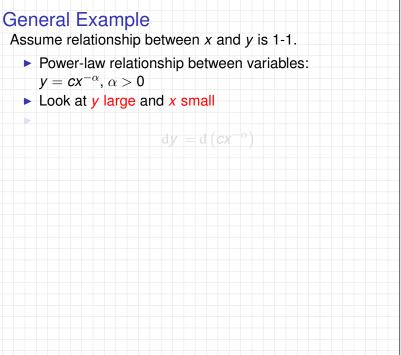
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Power-Law Mechanisms

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Assume relationship between x and y is 1-1.

Power-law relationship between variables:

$$y = cx^{-lpha}, \, \alpha > 0$$

Þ

Look at y large and x small

$$\mathrm{d}y = \mathrm{d}\left(\mathbf{c}x^{-\alpha}\right)$$

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$$\mathrm{d}x = \frac{-c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y$$

Power-Law Mechanisms

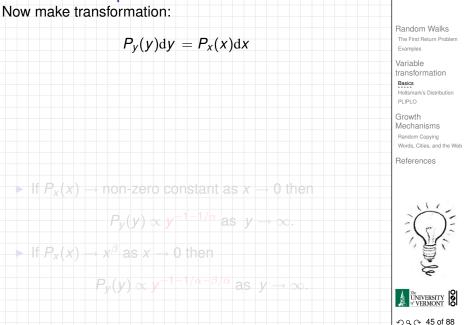
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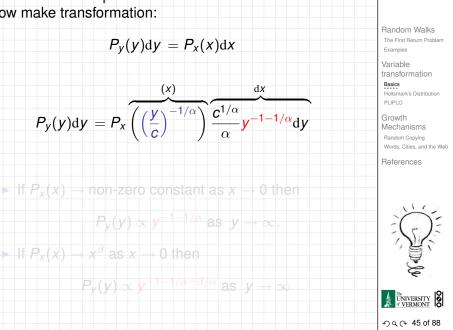




Power-Law

Mechanisms

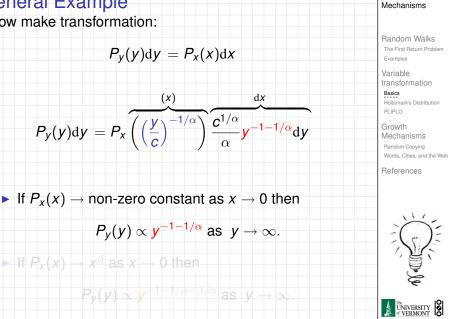
Now make transformation:



Power-Law

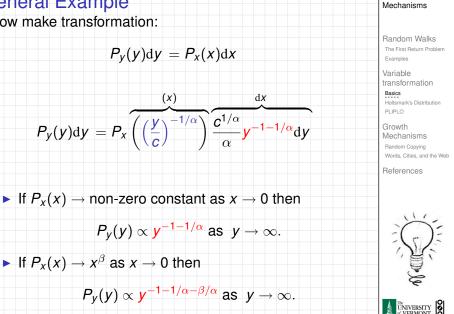
Mechanisms

Now make transformation:



Power-Law

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Power-Law

Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda} e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/lpha} + O\left(y^{-1-2/lpha}\right)$$

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Exponentials arise from randomness...

Power-Law Mechanisms

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More later when we cover robustness.

Power-Law Mechanisms

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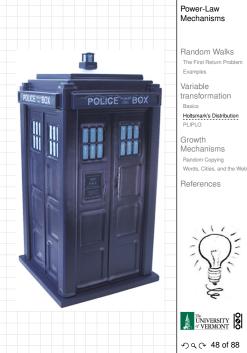
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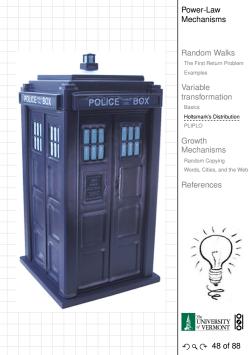
Outline	Power-Law Mechanisms
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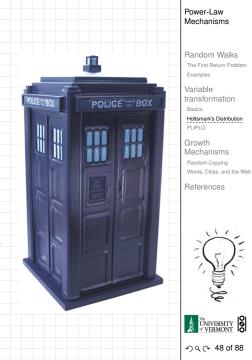
- Select a random point in the universe \vec{x}
- (possible all of space-time)
- Measure the force of gravity
 F(x)
- Observe that $P_F(F) \sim F^{-5/2}$.



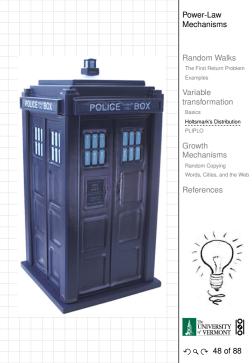
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Matter is concentrated in stars:

- F is distributed unevenly
- Probability of being a distance *r* from a single star at $\vec{x} = \vec{0}$:
 - $P_r(r)\mathrm{d}r\propto r^2\mathrm{d}r$
- Assume stars are distributed randomly in space (oops?)
- Assume only one star has significant effect at \vec{x} .
- Law of gravity:



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invert:

$$r \propto F^{-1/2}$$

Power-Law Mechanisms

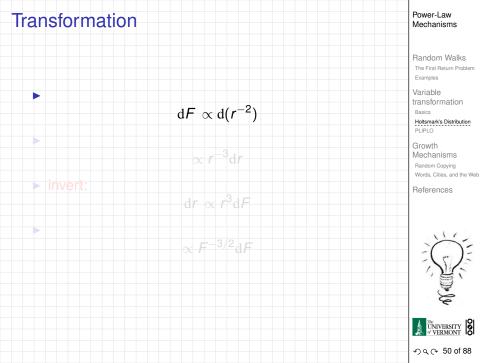
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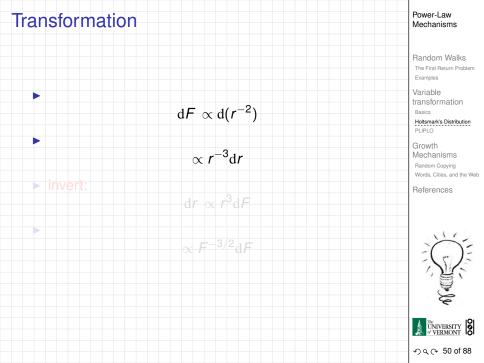
Variable transformation Basics

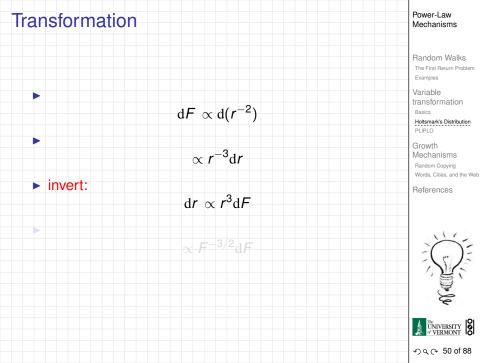
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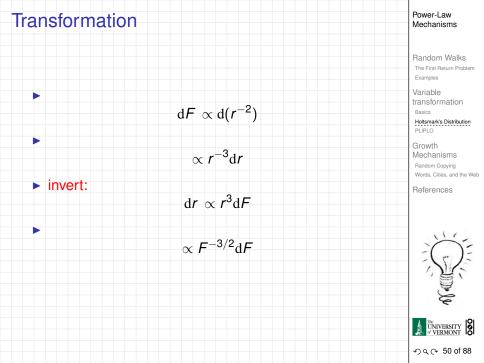
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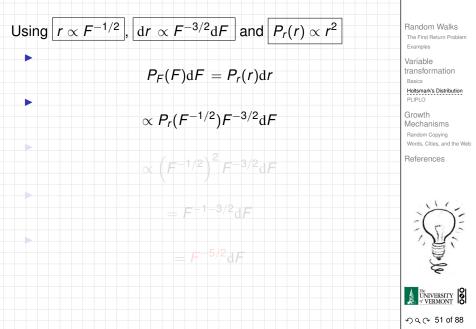
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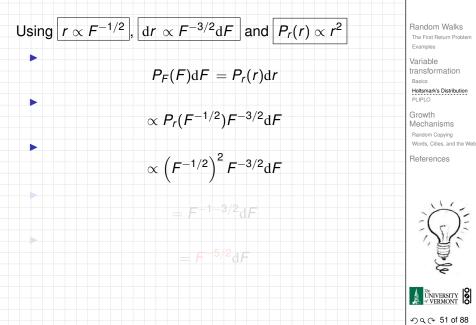


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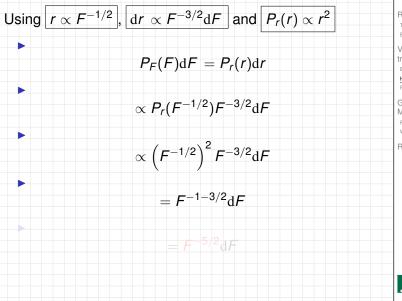
Using
$$r \propto F^{-1/2}$$
, $dr \propto F^{-3/2}dF$ and $P_r(r) \propto r^2$
 $P_F(F)dF = P_r(r)dr$
 $\propto P_r(F^{-1/2})F^{-3/2}dF$
 $\propto (F^{-1/2})^2 F^{-3/2}dF$
 $= F^{-1-3/2}dF$



Power-Law Mechanisms



Power-Law Mechanisms



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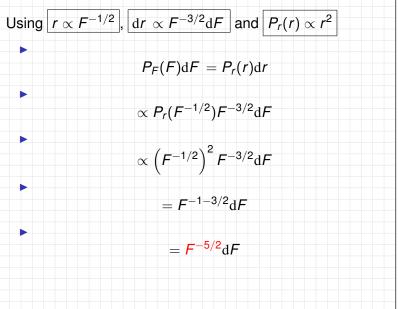
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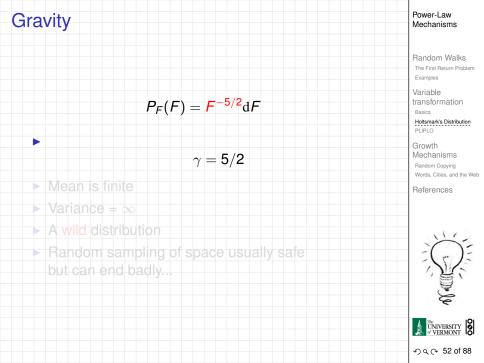
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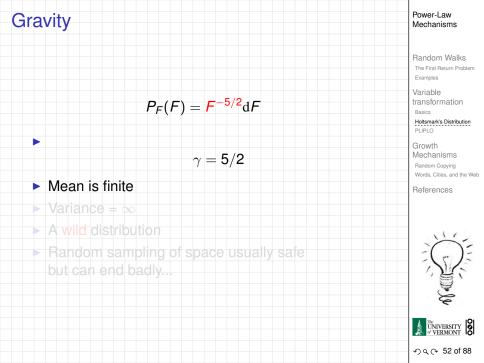
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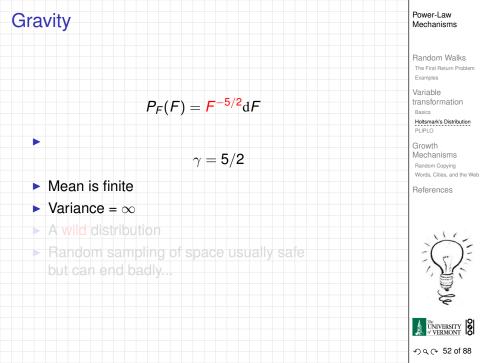
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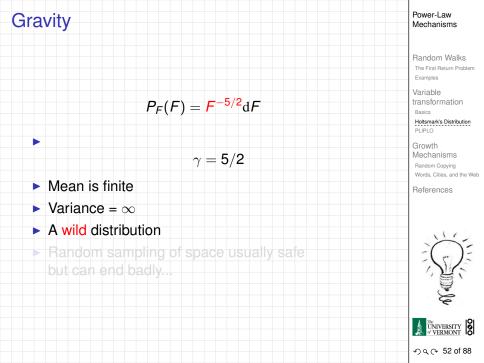


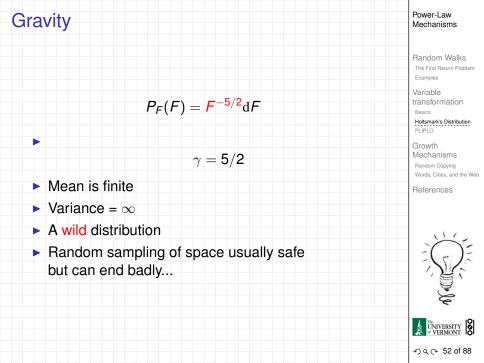
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Caution!

PLIPLO = Power law in, power law out

- Explain a power law as resulting from another unexplained power law.
- Yet another homunculus argument (E).
- Don't do this!!! (slap, slap)
- We need mechanisms!

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Random walks represent additive aggregation

- Mechanism: Random addition and subtraction
- Compare across realizations, no competition.
- Next: Random Additive/Copying Processes involving Competition.
- Widespread: Words, Cities, the Web, Wealth, Productivity (Lotka), Popularity (Books, People, ...)
- Competing mechanisms (trickiness

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Random Copying Words, Cities, and the Web



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1924: G. Udny Yule ^[14]: # Species per Genus

- 1926: Lotka^{[6}
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 - The World Wide Web, networks-at-large

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- ▶ 1965/1976: Derek de Solla Price ^{[9,}
 - Network of Scientific Citations
- 1999: Barabasi and Albert^[1]:
 - The World Wide Web, networks-at-large

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- 1924: G. Udny Yule ^[14]:
 # Species per Genus
- 1926: Lotka^[6]:
 - # Scientific papers per author (Lotka's law)
- 1953: Mandelbrot^[8]: Optimality argument for Zipf's law; focus on language.
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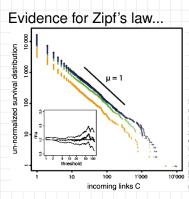
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Rasics Growth FIG. 1 (color online). (Color Online) Log-log plot of the number of packages in four Debian Linux Distributions with more than C in-directed links. The four Debian Linux Distributions are Woody (19.07.2002) (orange diamonds), Sarge (06.06.2005) (green crosses), Etch (15.08.2007) (blue circles), Lenny (15.12.2007) (black+'s). The inset shows the maximum likelihood estimate (MLE) of the exponent μ together with two boundaries defining its 95% confidence interval (approximately given by $1 \pm 2/\sqrt{n}$, where n is the number of data points using in the MLE), as a function of the lower threshold. The MLE has been modified from the standard Hill estimator to take into account the discreteness of C

Maillart et al., PRL, 2008: "Empirical Tests of Zipf's Law Mechanism in Open Source Linux Distribution"^[7]

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Essential Extract of a Growth Model Random Competitive Replication (RCR): 1. Start with 1 element of a particular flavor at t = 1

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Random Competitive Replication (RCR):

- 1. Start with 1 element of a particular flavor at t = 1
- 2. At time t = 2, 3, 4, ..., add a new element in one of two ways:
 - With probability ρ, create a new element with a new flavor
 - With probability 1 ρ, randomly choose from all existing elements, and make a copy.
 - Elements of the same flavor form a group

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Random Competitive Replication (RCR):

- 1. Start with 1 element of a particular flavor at t = 1
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 - Mutation/Innovation
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 - With probability 1 ρ, randomly choose from all existing elements, and make a copy.
 Replication/Imitation
 - Elements of the same flavor form a group

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Example: Words in a text

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Consider words as they appear sequentially.
 With probability ρ, the next word has not previously appeared

With probability $1 - \rho$, randomly choose one word from all words that have come before, and reuse this

word

Example: Words in a text

- Consider words as they appear sequentially.
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 With probability 1 - ρ, randomly choose one word from all words that have come before, and reuse this word Power-Law Mechanisms

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Competition for replication between elements is random

- Competition for growth between groups is not random
- Selection on groups is biased by size
- Rich-gets-richer story
- Random selection is easy
- No great knowledge of system needed

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Steady growth of system: +1 element per unit time.

- Steady growth of distinct flavors at rate r
- We can incorporate
 - Elements moving between groups

 - 4 Ditterent selection dated on group bite

Steady growth of system: +1 element per unit time.

Steady growth of distinct flavors at rate ρ

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- Steady growth of system: +1 element per unit time.
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- We can incorporate
 - 1. Element elimination
 - 2. Elements moving between groups
 - 3. Variable innovation rate /
 - 4. Different selection based on group size

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Steady growth of system: +1 element per unit time.

- Steady growth of distinct flavors at rate ρ
- We can incorporate
 - 1. Element elimination
 - 2. Elements moving between groups
 - 3. Variable innovation rate ρ
 - Different selection based on group size (But mechanism for selection is not as simple...)



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Definitions:

- k_i = size of a group i
- \triangleright $N_k(t) = #$ groups containing k elements at time t.

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Definitions:

- k_i = size of a group i
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Basic question: How does $N_k(t)$ evolve with time?

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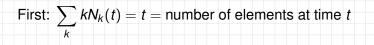


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$P_k(t)$ = Probability of choosing an element that belongs to a group of size *k*:

- N_k(t) size k groups
- $\blacktriangleright \Rightarrow kN_k(t)$ elements in size k groups
- t elements overall

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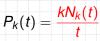
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 $N_k(t)$, the number of groups with k elements, changes at time t if

 An element belonging to a group with k elements i replicated

 An element belonging to a group with k – 1 elements is replicated Power-Law Mechanisms

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 An element belonging to a group with k – 1 elements is replicated Power-Law Mechanisms

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 $N_k(t)$, the number of groups with k elements, changes at time t if

 An element belonging to a group with k elements is replicated

2. An element belonging to a group with k - 1 elements is replicated

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 $N_k(t)$, the number of groups with k elements, changes at time t if

1. An element belonging to a group with k elements is replicated $N_k(t+1) = N_k(t) - 1$

2. An element belonging to a group with k - 1 elements is replicated

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 $N_k(t)$, the number of groups with k elements, changes at time t if

1. An element belonging to a group with k elements is replicated $N_k(t+1) = N_k(t) - 1$

Happens with probability $(1 - \rho)kN_k(t)/t$

2. An element belonging to a group with k - 1 elements is replicated

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Happens with probability $(1 - \rho)kN_k(t)/t$

An element belonging to a group with k – 1 elements is replicated

 $N_k(t+1) = N_k(t) + 1$

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 $N_k(t)$, the number of groups with k elements, changes at time t if

1. An element belonging to a group with k elements is replicated $N_k(t+1) = N_k(t) - 1$ Happens with probability $(1 - \rho)kN_k(t)/t$

2. An element belonging to a group with k - 1 elements is replicated $N_k(t+1) = N_k(t) + 1$

Happens with probability $(1 - \rho)(k - 1)N_{k-1}(t)/t$

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Special case for $N_1(t)$:

1. The new element is a new flavor:

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1. The new element is a new flavor:

Special case for $N_1(t)$:

2. A unique element is replicated.

Special case for $N_1(t)$:

- 1. The new element is a new flavor:
 - $N_1(t+1) = N_1(t) + 1$

2. A unique element is replicated.

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1. The new element is a new flavor: $N_1(t+1) = N_1(t) + 1$

Happens with probability ρ

Special case for $N_1(t)$:

2. A unique element is replicated.

Special case for $N_1(t)$:

- 1. The new element is a new flavor:
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 - Happens with probability ρ

2. A unique element is replicated. $N_1(t+1) = N_1(t) - 1$

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Special case for $N_1(t)$:

- 1. The new element is a new flavor:
 - $N_1(t+1) = N_1(t) + 1$ Happens with probability ρ
- 2. A unique element is replicated. $N_1(t+1) = N_1(t) - 1$ Happens with probability $(1 - \rho)N_1/t$

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Put everything together: For k > 1:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-
ho) \left((k-1) \frac{N_{k-1}(t)}{t} - k \frac{N_k}{t} \right)$$

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Put everything together: For k > 1:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho) \left((k-1) \frac{N_{k-1}(t)}{t} - k \frac{N_k}{t} \right)$$

For *k* = 1:

$$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)\mathbf{1} \cdot \frac{N_1(t)}{t}$$

. . . .

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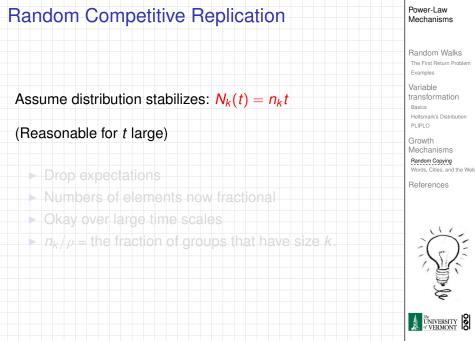
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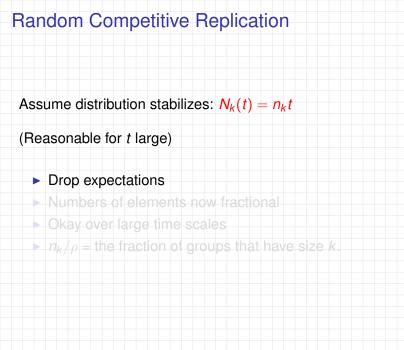
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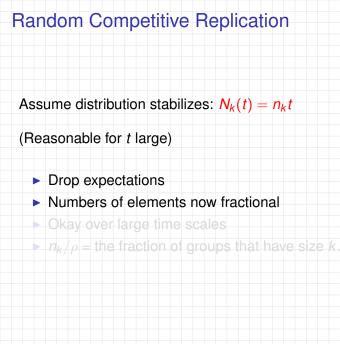
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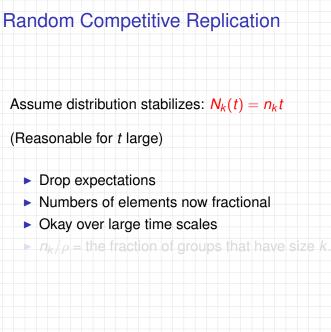
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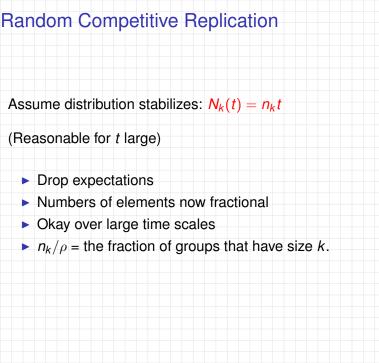
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Stochastic difference equation:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho) \left((k-1) \frac{N_{k-1}(t)}{t} - k \frac{N_k(t)}{t} \right)$$

becomes

$$n_k(t+1) - n_k t = (1-\rho)\left((k-1)\frac{n_{k-1}t}{t} - k\frac{n_k t}{t}\right)$$

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Stochastic difference equation:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho) \left((k-1) \frac{N_{k-1}(t)}{t} - k \frac{N_k(t)}{t} \right)$$

becomes

$$n_k(t+1) - n_k t = (1-\rho)\left((k-1)\frac{n_{k-1}t}{t} - k\frac{n_k t}{t}\right)$$

$$n_k(t+1-t) = (1-\rho)\left((k-1)\frac{n_{k-1}t}{t} - k\frac{n_kt}{t}\right)$$

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 $\Rightarrow n_k = (1 - \rho) \left((k - 1)n_{k-1} - kn_k \right)$

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 $\Rightarrow n_k (1 + (1 - \rho)k) = (1 - \rho)(k - 1)n_{k-1}$

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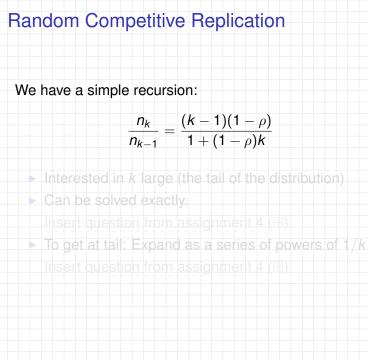
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We have a simple recursion:

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}$$

Interested in k large (the tail of the distribution)

- Can be solved exactly.
- Insert question from assignment 4 (⊞)
 To get at tail: Expand as a series of powers of 1/k

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Power-Law Mechanisms

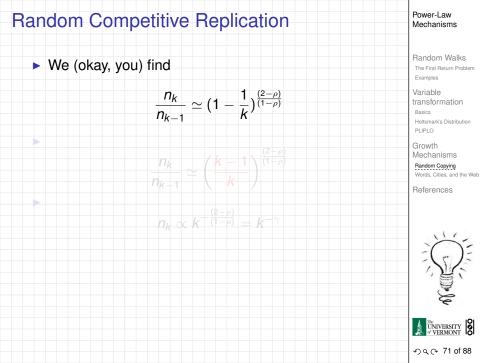
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Power-Law Random Competitive Replication Mechanisms Random Walks We (okay, you) find The First Return Problem Examples Variable $\frac{n_k}{n_{k-1}} \simeq (1 - \frac{1}{k})^{\frac{(2-\rho)}{(1-\rho)}}$ transformation Rasics Holtsmark's Distribution PLIPLO ► Growth $\frac{n_k}{n_{k-1}} \simeq \left(\frac{k-1}{k}\right)^{\frac{(2-\rho)}{(1-\rho)}}$ Mechanisms Random Copying Words, Cities, and the Web References 빌 UNIVERSITY 𝕎 𝔄 𝔅 71 of 88

Random Competitive Replication We (okay, you) find $\frac{n_k}{n_{k-1}} \simeq (1 - \frac{1}{k})^{\frac{(2-\rho)}{(1-\rho)}}$ ► $\frac{n_k}{n_{k-1}} \simeq \left(\frac{k-1}{k}\right)^{\frac{(2-\rho)}{(1-\rho)}}$ ► $n_k \propto k^{-rac{(2ho)}{(1ho)}} = k^{-\gamma}$

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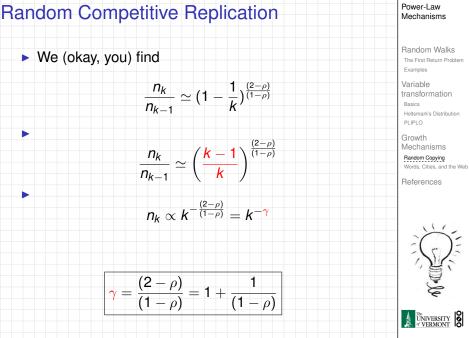
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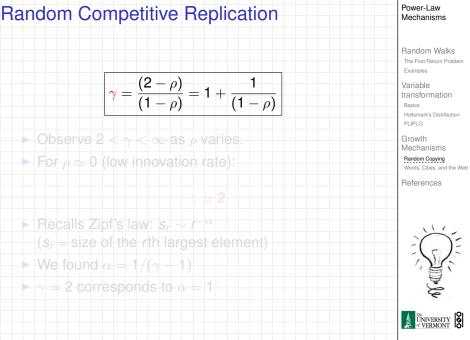
References



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 $\gamma = rac{(2ho)}{(1ho)} = 1 + rac{1}{(1ho)}$

• Observe 2 < γ < ∞ as ρ varies.

For $\rho \simeq 0$ (low innovation rate):

~ 2

- Recalls Zipf's law: $s_r \sim r^{-\alpha}$
 - $(s_r = size of the rth largest element)$
- We found $\alpha = 1/(\gamma 1)$
- > $\gamma = 2$ corresponds to $\alpha = 1$

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Power-Law Mechanisms

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We (roughly) see Zipfian exponent^[15] of α = 1 for many real systems: city sizes, word distributions, ...

Corresponds to $\rho \rightarrow 0$ (Krugman doesn't like it)^[5]

- But still other mechanisms are possible...
- Must look at the details to see if mechanism makes sense...

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- Must look at the details to see if mechanism makes sense... more later.

We had one other equation:

►

$$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)\mathbf{1} \cdot \frac{N_1(t)}{t}$$

As before, set $N_1(t) = n_1 t$ and drop expectations

 $n_1(t+1) - n_1t = \rho - (1-\rho)1 \cdot \frac{n_1t}{t}$







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NI (1)

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$$n_1(t+1) - n_1t = \rho - (1-\rho)1 \cdot \frac{n_1}{2}$$

 $n_1 = \rho - (1 - \rho)n_1$

Rearrange:

Power-Law Mechanisms

NI (1)

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$$n_1 + (1-\rho)n_1 = \rho$$

Power-Law Mechanisms

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Rearrange:

$$n_1+(1-\rho)n_1=\rho$$

 $n_1 = \rho - (1 - \rho)n_1$

$$n_1 = \frac{\rho}{2-\rho}$$

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NI (1)

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So...

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to groups	of size 1):	
	$N_1(t)$	1
	ρt	$=$ $\frac{1}{2-\rho}$
(also = fra	ction of groups	of size 1)

Recall number of distinct elements = ρt .

- For ho small, fraction of unique elements $\sim 1/2$
- Roughly observed for real distributions
- \triangleright ρ increases, fraction increases

 $N_1(t) = n_1 t = \frac{\rho t}{2 - \rho}$

Fraction of distinct elements that are unique (belong)

Random Competitive Replication $N_1(t) = n_1 t = \frac{\rho t}{2 - \rho}$ So... • Recall number of distinct elements = ρt . Fraction of distinct elements that are unique (belong For ρ small, fraction of unique elements $\sim 1/2$ Roughly observed for real distributions \triangleright ρ increases, fraction increases

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So...
$$N_1(t) = n_1 t = \frac{\rho t}{2 - \rho}$$

- Recall number of distinct elements = pt.
- Fraction of distinct elements that are unique (belong to groups of size 1):

$$\frac{N_1(t)}{\rho t} = \frac{1}{2-t}$$

D

(also = fraction of groups of size 1)

- For ho small, fraction of unique elements \sim 1/2
- Roughly observed for real distributions
- $\triangleright \rho$ increases, fraction increases
- \triangleright Can show fraction of groups with two elements \sim 1
- Model does well at both ends of the distribution

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D

(also = fraction of groups of size 1)

- For ρ small, fraction of unique elements $\sim 1/2$
- Roughly observed for real distributions
- ρ increases, fraction increases
- Can show fraction of groups with two elements $\sim 1/6$
- Model does well at both ends of the distribution

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Random Competitive Replication

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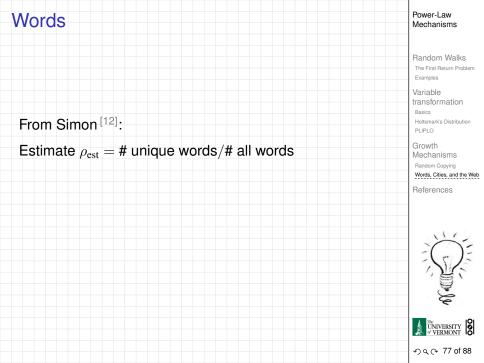
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Words

From Simon^[12]:

Estimate $\rho_{\rm est} = \text{\#}$ unique words/# all words

For Joyce's Ulysses: $\rho_{est} \simeq 0.115$

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Words				Power-Law Mechanisms
				Random Walks The First Return Problem Examples Variable transformation Basics
From Simor	ו ^[12] :			Holtsmark's Distribution PLIPLO
Estimate $ ho_{e}$ For Joyce's	Growth Mechanisms Random Copying Words, Cities, and the Web			
		N ₂ (real)	N ₂ (est)	References
	15,850	4,776	4,870	
				N. S. S.
				Vermont
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- Yule's paper (1924)^[14]:
 "A mathematical theory of evolution, based on the conclusions of Dr J. C. Willis, F.R.S."
- Simon's paper (1955) ^[12]:
 "On a class of skew distribution functions" (slipping)

From Simon's introduction



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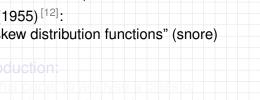
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From Simon's introduction:

It is the purpose of this paper to analyse a class of distribution functions that appear in a wide range of empirical data—particularly data describing sociologic biological and economoic phenomena.

Its appearance is so frequent, and the phenomena so diverse, that one is led to conjecture that if these phenomena have any property in common it can only b a similarity in the structure of the underlying probability mechanisms. Power-Law Mechanisms

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- Yule's paper (1924)^[14]:
 "A mathematical theory of evolution, based on the conclusions of Dr J. C. Willis, F.R.S."
- Simon's paper (1955) ^[12]:
 "On a class of skew distribution functions" (snore)

From Simon's introduction:

It is the purpose of this paper to analyse a class of distribution functions that appear in a wide range of empirical data—particularly data describing sociological, biological and economoic phenomena.

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More on Herbert Simon (1916-2001):

- Political scientist
- Involved in Cognitive Psychology, Computer Scienc Public Administration, Economics, Management, Sociology
- Coined 'bounded rationality' and 'satisficing'
- Nearly 1000 publications
- An early leader in Artificial Intelligence, Information Processing, Decision-Making, Problem-Solving, Attention Economics, Organization Theory, Complex Systems, And Computer Simulation Of Scientific Discovery.
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- Citation network of scientific papers
- Price's term: Cumulative Advantage
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- (Hath = unit of purchasing power.)
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"For to every one that hath shall be given...

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Merton was a catchphrase machine:

- 1. self-fulfilling prophecy
- 2. role model
- 3. unintended (or unanticipated) consequences
- focused interview --> focus group

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And just to be clear...

Merton's son, Robert C. Merton, won the Nobel Prize for Economics in 1997.

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- Another term: "Preferential Attachment"
- Considered undirected networks (not realistic but avoids 0 citation problem)
- Still have selection problem based on size (non-random)
- Solution: Randomly connect to a node (easy)
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