Lognormals and friends Principles of Complex Systems CSYS/MATH 300, Fall, 2010

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont



















Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan





Outline

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model Random Growth with Variable Lifespan

References

Lognormals

Empirical Confusability
Random Multiplicative Growth Model
Random Growth with Variable Lifespan





Outline

Lognormals and friends

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with

Variable Lifespan
References

Lognormals

Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan





Alternative distributions

Lognormals and friends

There are other 'heavy-tailed' distributions:

1. The Log-normal distribution (⊞)

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions (🗉

$$P(x)dx = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{\mu+1} e^{-(x/\lambda)^{\mu}} dx$$

CCDF = stretched exponential (III).

3. Gamma distributions (III), and more

Lognormals Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan





There are other 'heavy-tailed' distributions:

1. The Log-normal distribution (⊞)

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions (⊞)

$$P(x)dx = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}} dx$$

 $CCDF = stretched exponential (<math>\square$).

Gamma distributions (⊞), and more





There are other 'heavy-tailed' distributions:

1. The Log-normal distribution (⊞)

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

2. Weibull distributions (⊞)

$$P(x)dx = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{\mu-1} e^{-(x/\lambda)^{\mu}} dx$$

 $CCDF = stretched exponential (<math>\square$).

3. Gamma distributions (\boxplus), and more.





The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- In x is distributed according to a normal distribution with mean μ and variance σ .
- Appears in economics and biology where growth increments are distributed normally.



Lognormals

Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

For lognormals:

$$\mu_{
m lognormal}=e^{\mu+rac{1}{2}\sigma^2}, \qquad {
m median}_{
m lognormal}=e^{\mu},$$
 ${
m normal}=(e^{\sigma^2}-1)e^{2\mu+\sigma^2}, \qquad {
m mode}_{
m lognormal}=e^{\mu-\sigma^2}.$

► All moments of lognormals are finite.



$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

For lognormals:

$$\mu_{ ext{lognormal}} = e^{\mu + rac{1}{2}\sigma^2}, \qquad ext{median}_{ ext{lognormal}} = e^{\mu},$$
 $\sigma_{ ext{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \qquad ext{mode}_{ ext{lognormal}} = e^{\mu - \sigma^2}$

All moments of lognormals are finite.

Lognormals Empirical Confusability

Random Multiplicative Growth Model

Variable Lifespan







$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

For lognormals:

$$\mu_{ ext{lognormal}} = e^{\mu + rac{1}{2}\sigma^2}, \qquad ext{median}_{ ext{lognormal}} = e^{\mu},$$
 $\sigma_{ ext{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \qquad ext{mode}_{ ext{lognormal}} = e^{\mu - \sigma^2}.$

► All moments of lognormals are finite.

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with

Variable Lifespan
References





Take Y as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}} dy \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set
$$Y = \ln X$$

Lognormals and friends

Lognormals Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan







Take Y as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma}dy \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln \lambda$

Lognormals and friends

Lognormals Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan





Take *Y* as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}}dy \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

Transform according to P(x)dx = P(y)dy.

$$\frac{1}{2} \exp\left(-\frac{(\ln x - \mu)^2}{2}\right) dx$$

Lognormals and friends

Lognormals Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan







Take Y as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma}dy \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

► Transform according to P(x)dx = P(y)dy:

$$\frac{dx}{dx} = 1/x \Rightarrow dy = dx/x$$

$$\Rightarrow P(x)dx = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

Lognormals and friends

Lognormals Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan





Take Y as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi\sigma}}dy \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

► Transform according to P(x)dx = P(y)dy:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$

$$\Rightarrow P(x)dx$$

$$2\pi\sigma$$

$$\frac{(\ln x - \mu)^2}{2\sigma^2}$$

Lognormals and friends

Lognormals Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan





Take Y as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma}dy \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

- ► Transform according to P(x)dx = P(y)dy:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$



$$\Rightarrow P(x)dx = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

Lognormals and friends

Lognormals Empirical Confusability

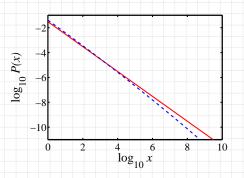
Random Multiplicative Growth Model

Variable Lifespan





Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude!

- For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- For power law (red), $\gamma = 1$ and c = 0.03.

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan





Lognormals and friends

What's happening:

$$\ln P(x) = \ln x$$

$$\times \sqrt{2\pi\sigma}$$
 $\times \sqrt{2}$

$$= \ln x - \ln \sqrt{2\pi}$$

$$\Rightarrow$$
 If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$



What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

 $\ln x + \ln \sqrt{2\pi} + (\ln x - \mu)^2$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right)$$

$$ightharpoonup$$
 \Rightarrow If $\sigma^2 \gg$ 1 and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

Lognormals and friends

Lognormals Empirical Confusability

Random Multiplicative Growth Model

Variable Lifespan





What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$



$$=-\ln x-\ln\sqrt{2\pi}-\frac{(\ln x-\mu)^2}{2\sigma^2}$$

$$\Rightarrow$$
 If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

Lognormals and friends

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model

Random Growth with Variable Lifespan





Lognormals and friends

What's happening:

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2} (\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right) \ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$

$$\Rightarrow$$
 If $\sigma^2 \gg$ 1 and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$





Lognormals and friends

What's happening:

Lognormals

• (1

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

Variable Lifespan
References

$$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$



$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right)\ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$

▶ \Rightarrow If $\sigma^2 \gg 1$ and μ ,

 $\ln P(x) \sim -\ln x + \text{const.}$



Lognormals and friends

What's happening:

Lognormals Empirical Confusability

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with

Variable Lifespan
References

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2} (\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1\right) \ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$

▶ \Rightarrow If $\sigma^2 \gg 1$ and μ ,

 $\ln P(x) \sim -\ln x + \text{const.}$





- Expect -1 scaling to hold until (ln x)² term becomes significant compared to (ln x).
- This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2} - 1\right)\ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} \epsilon$$

$$\simeq 0.05(\sigma^2 - \mu$$

⇒ If you find a -1 exponent,
you may have a lognormal distribution...

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable Lifespan





- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$.
- This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1\right) \ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$

$$\simeq 0.05(\sigma^2 - \mu)$$

→ If you find a -1 exponent,
you may have a lognormal distribution...

Lognormals and friends

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model

Random Growth with Variable Lifespan





- Expect -1 scaling to hold until (ln x)² term becomes significant compared to (ln x).
- This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2}-1\right)\ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} \epsilon$$

$$\simeq 0.05(\sigma^2 - \mu)$$

▶ If you find a -1 exponent, you may have a lognormal distribution..

Lognormals and friends

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model

Random Growth with Variable Lifespan





- Expect -1 scaling to hold until (ln x)² term becomes significant compared to (ln x).
- ► This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1\right) \ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$

$$\simeq 0.05(\sigma^2 - \mu)$$

→ If you find a -1 exponent,
you may have a lognormal distribution...

Lognormals and friends

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model

Random Growth with Variable Lifespan





- Expect -1 scaling to hold until (ln x)² term becomes significant compared to (ln x).
- This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2}-1\right)\ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$

$$\simeq 0.05 (\sigma^2 - \mu)$$

▶ If you find a -1 exponent, you may have a lognormal distribution..

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative

Growth Model

Random Growth with Variable Lifespan





- Expect -1 scaling to hold until (ln x)² term becomes significant compared to (ln x).
- This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05\left(\frac{\mu}{\sigma^2}-1\right)\ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$

$$\simeq 0.05 (\sigma^2 - \mu)$$

→ If you find a -1 exponent, you may have a lognormal distribution...





Outline

Lognormals Empirical Confusability Random Multiplicative Growth

Lognormals and

friends

Random Growth with Variable Lifespan References

Lognormals

Empirical Confusability

Random Multiplicative Growth Model







Lognormals Empirical Confusability Rendern Multiplicative Con

Lognormals and friends

Random Multiplicative Growth Random Growth with Variable Lifespan

References

Random multiplicative growth:

- $x_{n+1} = rx_n$
- where r > 0 is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:

$$\ln x_{n+1} = \ln r + \ln x_n$$

- \rightarrow ln x_n is normally distributed
- $\Rightarrow x_n$ is lognormally distributed





Lognormals

Lognormals and friends

Empirical Confusability

Random Multiplicative Growth

Random Growth with Variable Lifespan

References

Random multiplicative growth:

 $x_{n+1} = rx_n$

where r > 0 is a random growth variable

- (Shrinkage is allowed)
- In log space, growth is by addition

$$\ln x_{n+1} = \ln r + \ln x_n$$

- \rightarrow ln x_n is normally distributed
- $ightharpoonup \Rightarrow x_n$ is lognormally distributed





Random multiplicative growth:

-

$$x_{n+1} = rx_n$$

where r > 0 is a random growth variable

- (Shrinkage is allowed)
- ▶ In log space, growth is by addition:

$$\ln x_{n+1} = \ln r + \ln x_n$$

- \rightarrow $\ln x_n$ is normally distributed
- $\Rightarrow x_n$ is lognormally distributed



Lognormals
Empirical Confusability

Random Multiplicative Growth Random Growth with Variable Lifespan





Random multiplicative growth:

$$x_{n+1} = rx_n$$

where r > 0 is a random growth variable

- (Shrinkage is allowed)
- In log space, growth is by addition:

$$\ln x_{n+1} = \ln r + \ln x_n$$

- ightharpoonup \Rightarrow In x_n is normally distributed
- $\rightarrow x_n$ is lognormally distributed

Lognormals and friends

Lognormals Empirical Confusability

Random Multiplicative Growth Random Growth with Variable Lifespan





Random multiplicative growth:

$$x_{n+1} = rx_n$$

where r > 0 is a random growth variable

- (Shrinkage is allowed)
- ▶ In log space, growth is by addition:

$$\ln x_{n+1} = \ln r + \ln x_n$$

- ightharpoonup \Rightarrow In x_n is normally distributed
- $ightharpoonup \Rightarrow x_n$ is lognormally distributed



Lognormals Empirical Confusability

Random Multiplicative Growth Random Growth with Variable Lifespan





Lognormals or power laws?

- ► Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- ► But Robert Axtell (2001) shows a power law fits
- Problem of data censusing (missing small firms).

One mechanistic piece in Gibrat's model seems okay

Lognormals and friends

Lognormals Empirical Confusability

Bandom Multiplicative Growtl Random Growth with Variable Lifesnan







- ► Gibrat ^[2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- ▶ But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma =$ 2, not $\gamma =$ 1 (!)
- Problem of data censusing (missing small firms).

One mechanistic piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Random Growth with Variable Lifespan





- ▶ Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- ▶ But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).

 One mechanistic piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth

Random Growth with

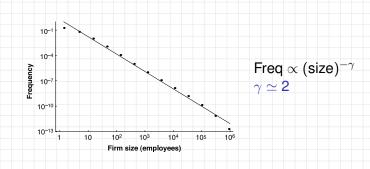
Variable Lifespan







- ▶ Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- ▶ But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma =$ 2, not $\gamma =$ 1 (!)
- Problem of data censusing (missing small firms).



 One mechanistic piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size Lognormals and friends

Lognormals

Empirical Confusability

Bandom Multiplicative Growth

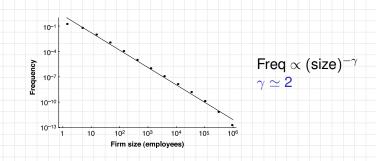
Random Growth with Variable Lifespan







- ▶ Gibrat [2] (1931) uses preceding argument to explain lognormal distribution of firm sizes ($\gamma \simeq 1$).
- ▶ But Robert Axtell [1] (2001) shows a power law fits the data very well with $\gamma = 2$, not $\gamma = 1$ (!)
- Problem of data censusing (missing small firms).



 One mechanistic piece in Gibrat's model seems okay empirically: Growth rate r appears to be independent of firm size. [1]. Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth

Bandom Growth with

Variable Lifespan







An explanation

- Axtel (mis?)cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq 1$
- ▶ The set up: N entities with size $x_i(t)$
- ► Generally

$$x_i(t+1) = rx_i(t)$$

- Same as for lognormal but one extra piece.
- ► Each x_i cannot drop too low with respect to the other

$$x_i(t+1) = \max(rx_i(t), c\langle x_i \rangle)$$



Lognormals and

friends

Lognormals
Empirical Confusability

Bandom Multiplicative Growth

Random Growth with Variable Lifespan





An explanation

- Axtel (mis?)cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq$ 1
- ▶ The set up: N entities with size $x_i(t)$
- ► Generally

$$x_i(t+1) = rx_i(t)$$

▶ Same as for lognormal but one extra piece.

► Each x_i cannot drop too low with respect to the other

$$x_i(t+1) = \max(rx_i(t), c(x_i))$$

friends

Lognormals Empirical Confusability

Lognormals and

Random Multiplicative Growth Random Growth with Variable Lifespan





- ▶ The set up: N entities with size $x_i(t)$
- ▶ Generally:

$$x_i(t+1)=rx_i(t)$$

where r is drawn from some happy distribution

- Same as for lognormal but one extra piece
- ► Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(rx_i(t), c(x_i))$$

Lognormals

Empirical Confusability

Random Multiplicative Growth Random Growth with Variable Lifespan





An explanation

- Axtel (mis?)cites Malcai et al.'s (1999) argument [5] for why power laws appear with exponent $\gamma \simeq$ 1
- ▶ The set up: N entities with size $x_i(t)$
- ▶ Generally:

$$x_i(t+1) = rx_i(t)$$

where r is drawn from some happy distribution

- Same as for lognormal but one extra piece.
- ► Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(rx_i(t), c\langle x_i \rangle)$$

Lognormals and friends

Lognormals Empirical Confusability

Random Multiplicative Growth Random Growth with Variable Lifespan





- ► The set up: N entities with size $x_i(t)$
- Generally:

$$x_i(t+1)=rx_i(t)$$

where r is drawn from some happy distribution

- Same as for lognormal but one extra piece.
- ► Each *x_i* cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(rx_i(t), c\langle x_i\rangle)$$

Lognormals

Empirical Confusability

Random Multiplicative Growth
Random Growth with
Variable Lifespan





An explanation

Some math later... Insert question from assignment

• Groovy...
$$c \text{ small} \Rightarrow \gamma \simeq 2$$

Lognormals Empirical Confusability Random Multiplicative Growth Bandom Growth with

Lognormals and

friends

Variable Lifesnan References







An explanation Some math later... Insert question from assignment

6 (⊞)

Lognormals Empirical Confusability Random Multiplicative Growth

Lognormals and

friends

Bandom Growth with Variable Lifesnan

References

Find $P(x) \sim x^{-\gamma}$ ▶ where ¬ is implicitly given by

• Groovy... c small $\Rightarrow \gamma \simeq 2$

Lognormals Empirical Confusability

Random Multiplicative Growth Random Growth with Variable Lifeenan

References

Some math later... Insert question from assignment

Find $P(x) \sim x^{-\gamma}$

• where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma - 1} - 1}{(c/N)^{\gamma - 1} - (c/N)} \right]$$

N = total number of firms.



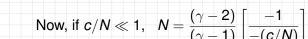


Find
$$P(x) \sim x^{-\gamma}$$

• where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma - 1} - 1}{(c/N)^{\gamma - 1} - (c/N)} \right]$$

N = total number of firms.









Lognormals and

Lognormals Empirical Confusability Random Multiplicative Growth Random Growth with Variable Lifeenan

References

friends

- Find $P(x) \sim x^{-\gamma}$ • where γ is implicitly given by
- $N = \frac{(\gamma 2)}{(\gamma 1)} \left[\frac{(c/N)^{\gamma 1} 1}{(c/N)^{\gamma 1} (c/N)} \right]$

$$N = \text{total number of firms.}$$

Now, if
$$c/N \ll 1$$
, $N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$

Which gives
$$\gamma \sim 1 + \frac{1}{1 - c}$$

Lognormals and

Lognormals Empirical Confusability Random Multiplicative Growth Random Growth with Variable Lifeenan

References

friends



References

friends

Lognormals and

Lognormals Empirical Confusability

Find $P(x) \sim x^{-\gamma}$

Bandom Multiplicative Growtl Random Growth with Variable Lifeenan

• where γ is implicitly given by

$$N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma - 1} - 1}{(c/N)^{\gamma - 1} - (c/N)} \right]$$

Now, if $c/N \ll 1$, $N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$

N = total number of firms.



Which gives $\gamma \sim 1 + \frac{1}{1-c}$





• Groovy... c small $\Rightarrow \gamma \simeq 2$

Outline

Lognormals Empirical Confusability

Growth Model

Lognormals and

friends

Random Multiplicative Random Growth with Variable References

Lognormals

- Empirical Confusability Random Multiplicative Growth Model
- Random Growth with Variable Lifespan





Ages of firms/people/... may not be the same

- Allow the number of updates for each size x; to vary
- Example: $P(t)dt = ae^{-at}dt$ where t = age.
- Back to no bottom limit: each x/ follows a lognormal
- ➤ Sizes are distributed as

Assume for this example that
$$\sigma \sim t$$
 and $u = \ln m$

Now averaging different lognormal distributions.

Lognormals and friends

Lognormals
Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with Variable





Lognormals and friends

Ages of firms/people/... may not be the same

- ightharpoonup Allow the number of updates for each size x_i to vary
- Example: $P(t)dt = ae^{-at}dt$ where t = age
- ► Back to no bottom limit: each x, follows a lognormal
- Sizes are distributed as

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

Now averaging different lognormal distributions.

Lognormals

Empirical Confusability

Random Multiplicative

Growth Model

Random Growth with Variable





Lognormals and friends

Ages of firms/people/... may not be the same

- ightharpoonup Allow the number of updates for each size x_i to vary
- **Example:** $P(t)dt = ae^{-at}dt$ where t = age.
- Back to no bottom limit: each x, follows a lognormal
- Sizes are distributed as

$$P(x) = \int_{t=0}^{\infty} \frac{dt}{dt} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

Now averaging different lognormal distributions.

Lognormals
Empirical Confusability
Random Multiplicative
Growth Model

Random Growth with Variable





Ages of firms/people/... may not be the same

- ightharpoonup Allow the number of updates for each size x_i to vary
- **Example**: $P(t)dt = ae^{-at}dt$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

Now averaging different lognormal distributions.

Lognormals and friends

Lognormals Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable





Ages of firms/people/... may not be the same

- ightharpoonup Allow the number of updates for each size x_i to vary
- **Example:** $P(t)dt = ae^{-at}dt$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- ► Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

Now averaging different lognormal distributions.

Lognormals and friends

Lognormals Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable





- ightharpoonup Allow the number of updates for each size x_i to vary
- **Example:** $P(t)dt = ae^{-at}dt$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [6]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

Now averaging different lognormal distributions.

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable





Averaging lognormals

Lognormals and friends

Lognormals
Empirical Confusability
Random Multiplicative
Growth Model

Random Growth with Variable
References

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) dt$$

- Insert question from assignment by
- Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$$







Averaging lognormals

friends

Lognormals and

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with Variable

Lognormals

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) dt$$

- ► Insert question from assignment 6 (⊞)
- Some enjoyable suffering leads to

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$$





Averaging lognormals

Lognormals and friends

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with Variable

References

Lognormals

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) dx$$

- ► Insert question from assignment 6 (⊞)
- Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$$





- $P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$
- Depends on sign of $\ln x/m$, i.e., whether x/m > 1 or x/m < 1.

$$x^{-1} - \sqrt{2\lambda}$$
 if x

- 'Break' in scaling (not uncommon)
- ► Double-Pareto distribution (⊞
- ► First noticed by Montroll and Shlesinger [7,8]
- Later: Huberman and Adamic [3, 4]: Number of pages



Lognormals and

friends

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with Variable







Lognormals and friends Lognormals

Empirical Confusability

- - $P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$
- ▶ Depends on sign of $\ln x/m$, i.e., whether x/m > 1 or x/m < 1.

$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1}-\sqrt{2\lambda} & \text{if } x/m > 1 \end{cases}$$

- ► 'Break' in scaling (not uncommon)
- Double-Pareto distribution (
- First noticed by Montroll and Shlesinger
- Later: Huberman and Adamic [3, 4]: Number of pages

Random Multiplicative Growth Model Random Growth with Variable References

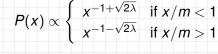






friends

- $P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$
- ▶ Depends on sign of $\ln x/m$, i.e., whether x/m > 1 or x/m < 1.



- 'Break' in scaling (not uncommon)

- Later: Huberman and Adamic [9, 4]: Number of pages



Lognormals and

Bandom Multiplicative Growth Model Random Growth with Variable

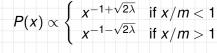






$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$

▶ Depends on sign of $\ln x/m$, i.e., whether x/m > 1 or x/m < 1.



- 'Break' in scaling (not uncommon)
- Double-Pareto distribution
- First noticed by Montroll and Shlesinger
- Later: Huberman and Adamic (3,4): Number of pages



Lognormals
Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with Variable







Lognormals and friends

•

- $P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$
- ▶ Depends on sign of $\ln x/m$, i.e., whether x/m > 1 or x/m < 1.
 - $P(x) \propto \left\{ egin{array}{ll} x^{-1+\sqrt{2\lambda}} & ext{if } x/m < 1 \ x^{-1-\sqrt{2\lambda}} & ext{if } x/m > 1 \end{array}
 ight.$
- 'Break' in scaling (not uncommon)
- ▶ Double-Pareto distribution (⊞)
- First noticed by Montroll and Shlesinger
- Later: Huberman and Adamic [3, 4]: Number of pages

Lognormals
Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with Variable







Lognormals and friends

•

- $P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$
- Depends on sign of $\ln x/m$, i.e., whether x/m > 1 or x/m < 1.
 - $P(x) \propto \left\{ egin{array}{ll} x^{-1+\sqrt{2\lambda}} & ext{if } x/m < 1 \ x^{-1-\sqrt{2\lambda}} & ext{if } x/m > 1 \end{array}
 ight.$
- 'Break' in scaling (not uncommon)
- ▶ Double-Pareto distribution (⊞)
- ► First noticed by Montroll and Shlesinger [7, 8]
- Later: Huberman and Adamic (3, 4): Number of pages

Lognormals
Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with Variable







$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$

Depends on sign of $\ln x/m$, i.e., whether x/m > 1 or x/m < 1.

$$P(x) \propto \left\{ egin{array}{ll} x^{-1+\sqrt{2\lambda}} & \mbox{if } x/m < 1 \ x^{-1-\sqrt{2\lambda}} & \mbox{if } x/m > 1 \end{array}
ight.$$

- 'Break' in scaling (not uncommon)
- ▶ Double-Pareto distribution (⊞)
- ► First noticed by Montroll and Shlesinger [7, 8]
- Later: Huberman and Adamic [3, 4]: Number of pages per website



Lognormals
Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with Variable





- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- ► Take-home message: Be careful out there...

Lognormals and friends

Lognormals
Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with Variable







- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take home message: Be careful out there.

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative

Growth Model

Random Growth with Variable





- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes the double Pareto distribution appears
- Take home message: Be careful out there.

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable







- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes,
 the double Pareto distribution appears
- Take home message: Be careful out there

Lognormals and friends

Lognormals

Empirical Confusability Random Multiplicative Growth Model

Random Growth with Variable





- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- ► Take-home message: Be careful out there...

Lognormals and friends

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable





References I

- [1] R. Axtell.

 Zipf distribution of U.S. firm sizes.

 Science, 293(5536):1818–1820, 2001. pdf (H)
- [2] R. Gibrat.
 Les inégalités économiques.
 Librairie du Recueil Sirey, Paris, France, 1931.
- [3] B. A. Huberman and L. A. Adamic. Evolutionary dynamics of the World Wide Web. Technical report, Xerox Palo Alto Research Center, 1999.
- [4] B. A. Huberman and L. A. Adamic. The nature of markets in the World Wide Web. Quarterly Journal of Economic Commerce, 1:5–12, 2000.

Lognormals and friends

Lognormals
Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan





References II

Lognormals and friends

[5] O. Malcai, O. Biham, and S. Solomon.
Power-law distributions and lévy-stable intermittent fluctuations in stochastic systems of many autocatalytic elements.
Phys. Rev. E, 60(2):1299–1303, 1999. pdf (H)

Lognormals
Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with

Variable Lifespan
References

[6] M. Mitzenmacher.

A brief history of generative models for power law and lognormal distributions.

Internet Mathematics, 1:226–251, 2003. pdf (⊞)

[7] E. W. Montroll and M. W. Shlesinger.
 On 1/f noise aned other distributions with long tails.
 Proc. Natl. Acad. Sci., 79:3380–3383, 1982. pdf (⊞)





References III

Lognormals and friends

Lognormals
Empirical Confusability

Random Multiplicative Growth Model Random Growth with Variable Lifespan

References

[8] E. W. Montroll and M. W. Shlesinger. Maximum entropy formalism, fractals, scaling phenomena, and 1/f noise: a tale of tails. J. Stat. Phys., 32:209–230, 1983.



