Positive Definite Matrices (PDMs)

# Lecture 26/28—Positive Definite Matrices

Linear Algebra MATH 124, Fall, 2010

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Lecture 26 Motivation... What a PDM is... Identifying PDMs Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem Nutshell Obtional material



# Outline

Lecture 26

Positive Definite Matrices (PDMs) Lecture 26 Motivation What a PDM is Identifying PDMs Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem Nutshell Optional material Motivation... What a PDM is... Identifying PDMs Completing the square ⇔ Gaussian elimination Principle Axis Theorem Nutshell **Optional material** 



# Outline

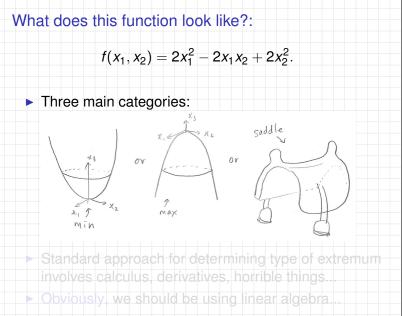


What does this function look like?:

 $f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2.$ 

Matrices (PDMs) Lecture 26 Motivation ... What a PDM is... Identifying PDMs Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem Nutshell Optional material UNIVERSITY 

Positive Definite



#### Positive Definite Matrices (PDMs)

Lecture 26 Motivation...

What a PDM is...

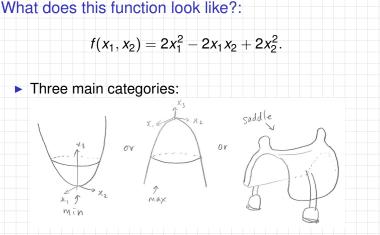
Identifying PDMs

Completing the square Gaussian elimination

Principle Axis Theorem

Nutshell





 Standard approach for determining type of extremum involves calculus, derivatives, horrible things...

Obviously, we should be using linear algebra...

Positive Definite Matrices (PDMs)

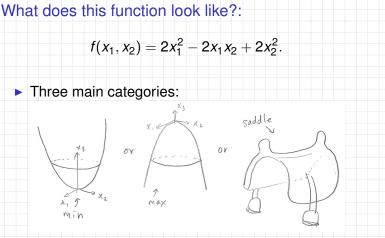
Lecture 26

Motivation... What a PDM is... Identifying PDMs Completing the square Gaussian elimination

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Positive Definite Matrices (PDMs)

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#### Linear Algebra-ization...

We can rewrite

$$f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2$$

Positive Definite Matrices (PDMs)

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Motivation... What a PDM is...

Identifying PDMs

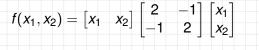
Completing the square  $\Leftrightarrow$ 

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Note: A is symmetric as A = A<sup>T</sup> (delicious).

Interesting and sneaky...



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Optional material

as

$$f(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \boxed{\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x}}$$

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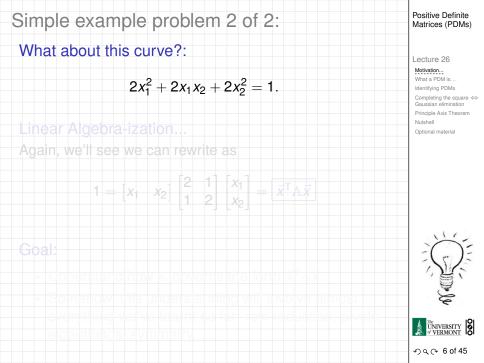
Optional material

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What about this curve?:  $2x_1^2 + 2x_1x_2 + 2x_2^2 = 1.$ Linear Algebra-ization... Again, we'll see we can rewrite as  $1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x}}$ 

Simple example problem 2 of 2:

Positive Definite Matrices (PDMs)

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Simple example problem 2 of 2: What about this curve?:  $2x_1^2 + 2x_1x_2 + 2x_2^2 = 1.$ Linear Algebra-ization... Again, we'll see we can rewrite as  $1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x}}$ Goal: • Understand how A governs the form  $\vec{x}^{T} \mathbb{A} \vec{x}$ .

 Somehow, this understanding will involve almost everything we've learnt so far: row reduction, pivol eigenthings, symmetry, ...

Positive Definite Matrices (PDMs)

Lecture 26 Motivation...

Identifying PDMs Completing the square Gaussian elimination Principle Avis Theorem

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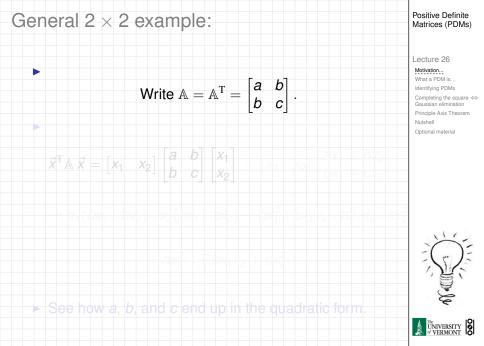
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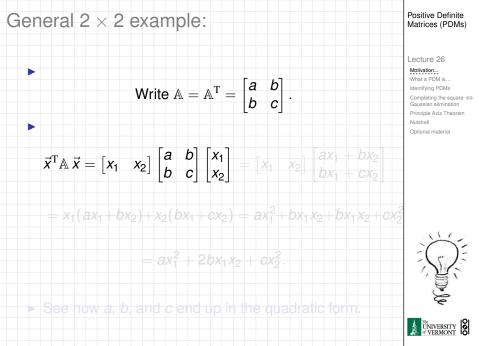
Positive Definite

Matrices (PDMs)





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General  $2 \times 2$  example:

►

Positive Definite Matrices (PDMs) Lecture 26 Motivation... What a PDM is... Write  $\mathbb{A} = \mathbb{A}^{\mathrm{T}} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ . Identifying PDMs Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem Nutshell Optional material  $\vec{x}^{\mathrm{T}} \mathbb{A} \, \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}$ 

 $= x_1(ax_1+bx_2)+x_2(bx_1+cx_2) = ax_1^2+bx_1x_2+bx_1x_2+cx_2^2$ 

See how a, b, and c end up in the quadratic form.



General  $2 \times 2$  example:

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Write  $\mathbb{A} = \mathbb{A}^{T} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ .  $\vec{x}^{T}\mathbb{A} \vec{x} = \begin{bmatrix} x_{1} & x_{2} \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_{1} \\ x_{2} \end{bmatrix} = \begin{bmatrix} x_{1} & x_{2} \end{bmatrix} \begin{bmatrix} ax_{1} + bx_{2} \\ bx_{1} + cx_{2} \end{bmatrix}$ 

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Positive Definite

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$$\mathbb{A} = \mathbb{A}^{\mathrm{T}} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$
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Identifying PDMs Completing the square Gaussian elimination Principle Axis Theorem Nutshell

What a PDM is...

Lecture 26 Motivation...

Positive Definite

Matrices (PDMs)

Optional material

$$\vec{x}^{\mathrm{T}} \mathbb{A} \, \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}$$

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 $ax_1 + 20x_1x_2 + 0x_2$ .

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 General 2 × 2 example:
 Positive Definite Matrices (PDMs)

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 Lecture 26

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 Completing the square  $\Leftrightarrow$  Gaussian elimination Principle Adis Theorem Nathelli Optical material

$$\vec{x}^{\mathrm{T}} \mathbb{A} \, \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}$$

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See how a, b, and c end up in the quadratic form.



Back to our first example:

We have:  $\vec{x}^{T} \wedge \vec{x} = ax_{1}^{2} + 2bx_{1}x_{2} + cx_{2}^{2} = f(x_{1}, x_{2})$ 

#### Positive Definite Matrices (PDMs)



Identify a = 2, b = -1, and c = 2.:f(x<sub>1</sub>, x<sub>2</sub>) = [x<sub>1</sub> x<sub>2</sub>]

2

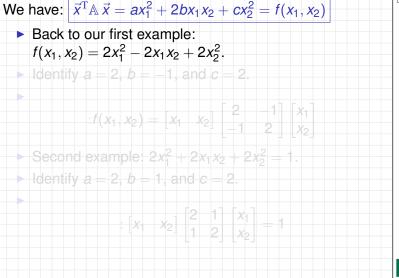
Second example:  $2x_1^2 + 2x_1x_2 + 2x_2^2 = 1.$ Identify a = 2, b = 1, and c = 2.

 $\begin{bmatrix} x_1 & x_2 \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & x_1 \\ 1 & 2 & x_2 \end{bmatrix} = 1$ 

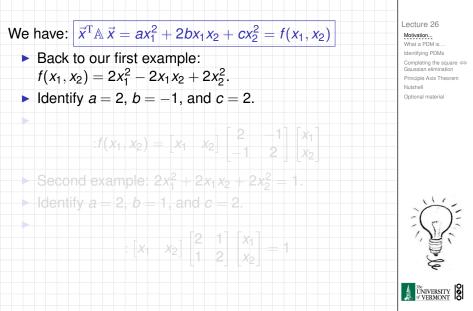


#### Positive Definite Matrices (PDMs)





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Positive Definite Matrices (PDMs)

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Back to our first example:

$$f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2$$

• Identify 
$$a = 2$$
,  $b = -1$ , and  $c = 2$ 

$$:f(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs Completing the square ⇔ Gaussian elimination Principle Axis Theorem Nutshell Optional material



We have: 
$$\vec{x}^{\mathrm{T}} \mathbb{A} \, \vec{x} = a x_1^2 + 2b x_1 x_2 + c x_2^2 = f(x_1, x_2)$$

• Back to our first example:  $f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2$ .

Identify 
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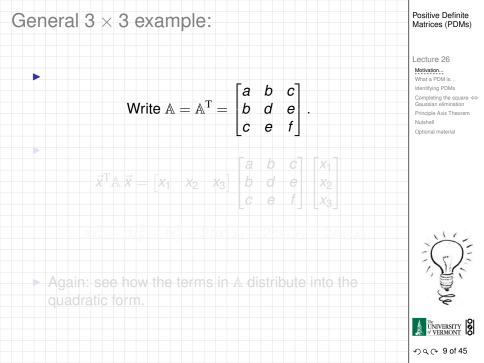
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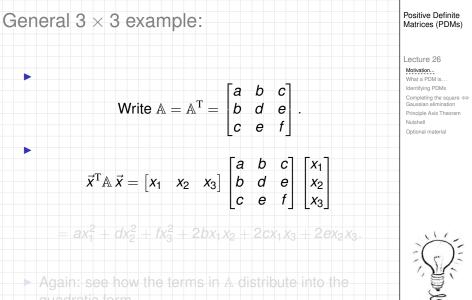
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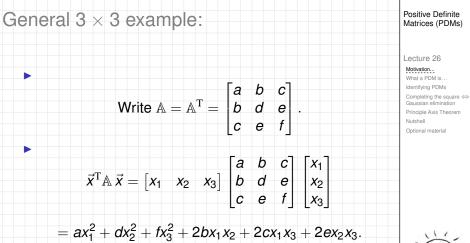






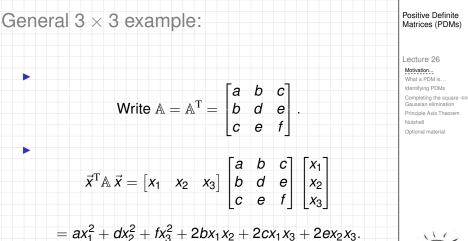
quadratic form.

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 Again: see how the terms in A distribute into the quadratic form.





 Again: see how the terms in A distribute into the quadratic form.



### General story:

#### Using the definition of matrix multiplication,



- We see the x<sub>i</sub>x<sub>i</sub> term is attached to a
- On diagonal terms look like this:  $a_{77}x^2$  and  $a_{33}x^2_3$ .
- Off-diagonal terms combine, e.g.,  $(a_{13} + a_{31}) \times x_3$ .
- Given some f with a term 23x<sub>1</sub>x<sub>3</sub>, we could divide the 23 between a<sub>13</sub> and a<sub>31</sub> however we like.
- e.g.,  $a_{13} = 36$  and  $a_{31} = -13$  would work.
  - But we choose to make A symmetric because symmetry is great.

#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation... What a PDM is...

Identifying PDMs

Completing the square  $\Leftrightarrow$  Gaussian elimination

Principle Axis Theorem

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#### General story:

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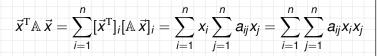
$$\vec{x}^{\mathrm{T}} \mathbb{A} \, \vec{x} = \sum_{i=1}^{n} [\vec{x}^{\mathrm{T}}]_i [\mathbb{A} \, \vec{x}]_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} x_j$$

We see the x,x, term is attached to a

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Motivation... What a PDM is...

Identifying PDMs

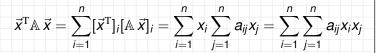
Completing the square  $\Leftrightarrow$  Gaussian elimination

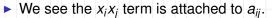
Principle Axis Theorem

Nutshell



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### Using the definition of matrix multiplication,

$$\vec{x}^{\mathrm{T}} \mathbb{A} \, \vec{x} = \sum_{i=1}^{n} [\vec{x}^{\mathrm{T}}]_{i} [\mathbb{A} \, \vec{x}]_{i} = \sum_{i=1}^{n} x_{i} \sum_{j=1}^{n} a_{ij} x_{j} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{i} x_{j}$$

# • We see the $x_i x_j$ term is attached to $a_{ij}$ .

• On-diagonal terms look like this:  $a_{77}x_7^2$  and  $a_{33}x_3^2$ .

#### Off-diagonal terms combine, e.g., (a13 + a31)x1x3.

- Given some f with a term 23x<sub>1</sub>x<sub>3</sub>, we could divide the 23 between a<sub>13</sub> and a<sub>31</sub> however we like.
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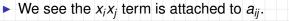
Principle Axis Theorem

Nutshell



#### Using the definition of matrix multiplication,

$$\vec{x}^{\mathrm{T}} \mathbb{A} \, \vec{x} = \sum_{i=1}^{n} [\vec{x}^{\mathrm{T}}]_{i} [\mathbb{A} \, \vec{x}]_{i} = \sum_{i=1}^{n} x_{i} \sum_{j=1}^{n} a_{ij} x_{j} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{i} x_{j}$$



- On-diagonal terms look like this:  $a_{77}x_7^2$  and  $a_{33}x_3^2$ .
- Off-diagonal terms combine, e.g.,  $(a_{13} + a_{31})x_1x_3$ .
- Given some f with a term 23x<sub>1</sub>x<sub>3</sub>, we could divide the 23 between a<sub>13</sub> and a<sub>31</sub> however we like.
- e.g.,  $a_{13} = 36$  and  $a_{31} = -13$  would work.
  - But we choose to make A symmetric because symmetry is great.

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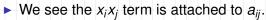
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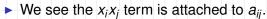
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### A few observations:

### 1. The construction $\vec{x}^{T} \mathbb{A} \vec{x}$ appears naturally.

- 2. Dimensions of  $\vec{x}^{T}$ ,  $\mathbb{A}$ , and  $\vec{x}$ 1 by *n*, *n* by *n*, and *n* by 1.
- 3.  $\vec{x}^{T} \wedge \vec{x}$  is a 1 by 1
- 4. If  $\mathbb{A}\vec{v} = \lambda\vec{v}$  then

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# Outline

Positive Definite Matrices (PDMs) Lecture 26 Motivation What a PDM is... Identifying PDMs Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem Lecture 26 Nutshell Optional material Motivation... What a PDM is... Identifying PDMs Principle Axis Theorem Nutshell UNIVERSITY

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### Positive Definite Matrices (PDMs):

Real, symmetric matrices with positive eigenvalues.

#### Math version:



## Semi-Positive Definite Matrices (SPDMs):



- $\lambda_1 \ge 0, \forall i = 1, 2, \cdots, n$
- Note: If admieleigenvalues are < 0 we have a st</p>

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### Positive Definite Matrices (PDMs):

- Real, symmetric matrices with positive eigenvalues.
- Math version:

$$\mathbb{A} = \mathbb{A}^{\mathrm{T}},$$
  
 $a_{ij} \in R \ \forall \ i, j = 1, 2, \cdots n,$   
and  $\lambda_i > 0, \ \forall \ i = 1, 2, \cdots n$ 

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$$\lambda_i \geq 0, \forall i = 1, 2, \cdots, n$$

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## Equivalent Definitions:

### Positive Definite Matrices:

• 
$$\mathbb{A} = \mathbb{A}^{\mathrm{T}}$$
 is a PDM if

$$\vec{x}^{\mathrm{T}} \mathbb{A} \, \vec{x} > \mathbf{0} \ \forall \, \vec{x} \neq \bar{\mathbf{0}}$$

### Semi-Positive Definite Matrices:

A = A is a SPDMi

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## Connecting these definitions:

## Spectral Theorem for Symmetric Matrices:

$$\mathbb{A} = \mathbb{Q} \, \Lambda \, \mathbb{Q}^{\mathrm{T}}$$

where  $\mathbb{Q}^{-1} = \mathbb{Q}^{\mathrm{T}}$ ,

$$\mathbb{Q} = \begin{bmatrix} \begin{vmatrix} & & & & & \\ \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_n \\ \mid & \mid & \cdots & \mid \end{bmatrix}, \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Special form of  $\mathbb{A} = \mathbb{S}\Lambda\mathbb{S}^{-1}$  that arises when  $\mathbb{A} = \mathbb{A}^{T}$ .

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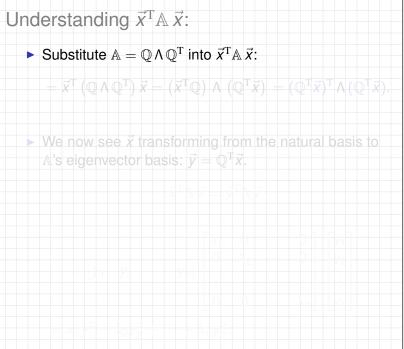
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- Substitute  $\mathbb{A} = \mathbb{Q} \wedge \mathbb{Q}^{\mathrm{T}}$  into  $\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x}$ :
  - $= \vec{x}^{\mathrm{T}} \left( \mathbb{Q} \wedge \mathbb{Q}^{\mathrm{T}} \right) \vec{x} = (\vec{x}^{\mathrm{T}} \mathbb{Q}) \wedge (\mathbb{Q}^{\mathrm{T}} \vec{x}) = (\mathbb{Q}^{\mathrm{T}} \vec{x})^{\mathrm{T}} \wedge (\mathbb{Q}^{\mathrm{T}} \vec{x}).$
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Understanding  $\vec{x}^{\mathrm{T}} \mathbb{A} \ \vec{x}$ :

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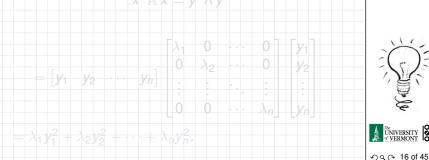


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$$= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} 0 & y_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_n \\ \vdots \\ y_n \end{bmatrix}$$
$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

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$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination

Principle Axis Theorem

Nutshell



#### So now we have ...

$$\vec{x}^{\mathrm{T}}\mathbb{A}\,\vec{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

- Can see whether or not x̄<sup>T</sup> A x̄ > 0 depends on the λ<sub>i</sub> since each y<sub>i</sub><sup>2</sup> > 0.
   So a PDM must have each λ<sub>i</sub> > 0.
- And a SPDM must have  $\lambda_i \ge 0$

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Lecture 26

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So now we have ...

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Lecture 26

What a PDM is...

Completing the square Gaussian elimination

Principle Axis Theorem

Nutshell



# More understanding of $\vec{x}^{T} \mathbb{A} \vec{x}$ :

Substitute  $\mathbb{A} = \mathbb{L} \mathbb{D} \mathbb{L}^{T}$  into  $\vec{x}^{T} \mathbb{A} \vec{x}$ :  $\vec{\mathbf{x}}^{\mathrm{T}}$  ( $\mathbf{L}$   $\mathbf{D}$   $\mathbf{L}^{\mathrm{T}}$ )  $\vec{\mathbf{x}}$  = ( $\vec{\mathbf{x}}^{\mathrm{T}}$  $\mathbf{L}$ )  $\mathbf{D}$  ( $\mathbf{L}^{\mathrm{T}}$  $\vec{\mathbf{x}}$ ) = ( $\mathbf{L}^{\mathrm{T}}$  $\vec{\mathbf{x}}$ )  $\mathbf{T}$   $\mathbf{D}$  ( $\mathbf{L}^{\mathrm{T}}$  $\vec{\mathbf{x}}$ ) • Change from eigenvalue story:  $\vec{x}$  is transformed into

#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$ 

Principle Axis Theorem

Nutshell

Optional material



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More understanding of  $\vec{x}^{T} \mathbb{A} \vec{x}$ :

- Substitute  $\mathbb{A} = \mathbb{L} \mathbb{D} \mathbb{L}^{\mathrm{T}}$  into  $\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x}$ :
  - $= \vec{x}^{\mathrm{T}} \left( \mathbb{L} \mathbb{D} \mathbb{L}^{\mathrm{T}} \right) \vec{x} = \left( \vec{x}^{\mathrm{T}} \mathbb{L} \right) \mathbb{D} \left( \mathbb{L}^{\mathrm{T}} \vec{x} \right) = \left( \mathbb{L}^{\mathrm{T}} \vec{x} \right)^{\mathrm{T}} \mathbb{D} \left( \mathbb{L}^{\mathrm{T}} \vec{x} \right)$

Change from eigenvalue story:  $\vec{x}$  is transformed i  $\vec{z} = \mathbb{L}^T \vec{x}$  but this is not a change of basis.





Substitute  $\mathbb{A} = \mathbb{L} \mathbb{D} \mathbb{L}^{\mathrm{T}}$  into  $\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x}$ :

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Change from eigenvalue story:  $\vec{x}$  is transformed in  $\vec{z} = \mathbb{L}^T \vec{x}$  but this is not a change of basis.



Gaussian elimination

Principle Axis Theorem Nutshell Optional material



Substitute  $\mathbb{A} = \mathbb{L} \mathbb{D} \mathbb{L}^{\mathrm{T}}$  into  $\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x}$ :

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#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$  Gaussian elimination

Principle Axis Theorem

Nutshell

Optional material

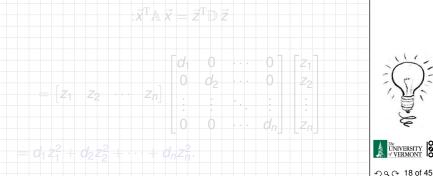


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Substitute 
$$\mathbb{A} = \mathbb{L} \mathbb{D} \mathbb{L}^{\mathrm{T}}$$
 into  $\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x}$ 

$$=ec{x}^{ ext{T}}\left(\mathbb{L}\,\mathbb{D}\,\mathbb{L}^{ ext{T}}
ight)ec{x}=\left(ec{x}^{ ext{T}}\mathbb{L}
ight)\,\mathbb{D}\,\left(\mathbb{L}^{ ext{T}}ec{x}
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Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

What a PDM is...

Completing the square  $\Leftrightarrow$  Gaussian elimination

Principle Axis Theorem

Nutshell

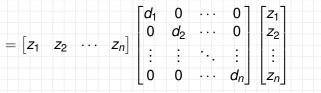


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#### Positive Definite Matrices (PDMs)

Lecture 26

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What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination

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Principle Axis Theorem

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#### So now we have ...

$$\vec{x}^{\mathrm{T}}\mathbb{A}\,\vec{x}=d_1z_1^2+d_2z_2^2+\cdots+d_nz_n^2$$

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Lecture 26

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Principle Axis Theorem

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Positive Definite Matrices (PDMs)

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Lecture 26

What a PDM is...

Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination

Principle Axis Theorem

Nutshell



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Lecture 26

Motivation...

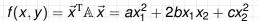
What a PDM is... Identifying PDMs

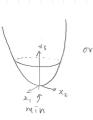
Completing the square  $\Leftrightarrow$ Gaussian elimination

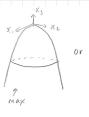
Principle Axis Theorem

Nutshell





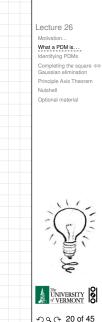




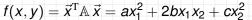


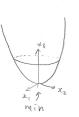
#### Focus on eigenvalues - We can now see:

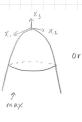
- $\frac{f(x,y)}{1.5.16} = \frac{1}{2} + \frac{1}$
- Niaximum ii 🖓 🗠 🗘 and 🖓 < 🛈
- Saddie: lif X > 0 and X < 0.</li>



Positive Definite Matrices (PDMs)









#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

What a PDM is...

Completing the square  $\Leftrightarrow$ Gaussian elimination

Principle Axis Theorem

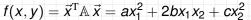
Nutshell

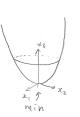
Optional material

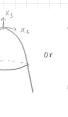
#### Focus on eigenvalues—We can now see:

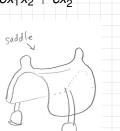
- f(x, y) has a minimum at x = y = 0 iff A is a PDM, i.e., if λ<sub>1</sub> > 0 and λ<sub>2</sub> > 0.
- Maximum: if  $\lambda_1 < 0$  and  $\lambda_2 < 0$ .
- Saddle: if  $\lambda_1 > 0$  and  $\lambda_2 < 0$ .











#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

What a PDM is...

Completing the square  $\Leftrightarrow$ Gaussian elimination

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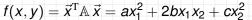
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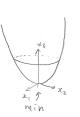
Max

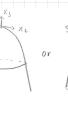
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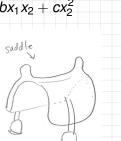
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#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

What a PDM is...

Completing the square Gaussian elimination

Principle Axis Theorem

Nutshell

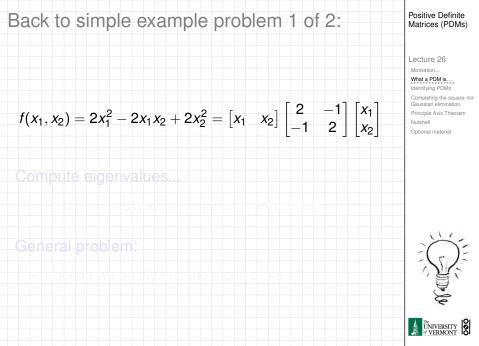
Optional material

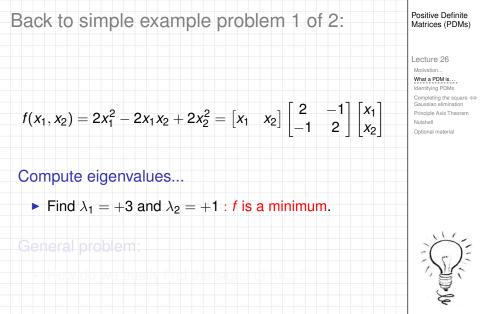
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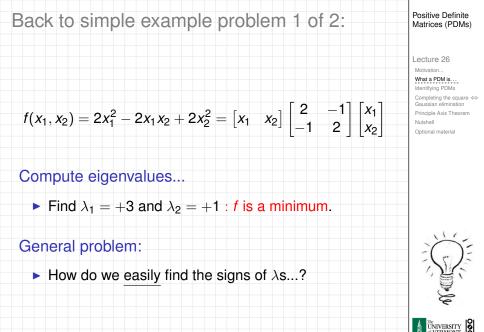






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# Outline

Lecture 26

Motivation...

Nutshell

Identifying PDMs

Principle Axis Theorem

Positive Definite Matrices (PDMs) Lecture 26 Motivation What a PDM is Identifying PDMs Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem Nutshell Optional material UNIVERSITY

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- We recall with alacrity the totally amazing fact that real symmetric matrices always have (1) real eigenvalues, and (2) orthogonal eigenvectors forming a basis for R<sup>n</sup>.
- We now see that knowing the signs of the λs is also important...

Test cases:



Some minor struggling leads to::

- A. P. L. = +3, X. = +1, (PDM, baob)
- $\triangleright$   $A_0: \lambda_1 = \pm \sqrt{5} \lambda_2 = -\sqrt{5} (sad)$
- $\blacktriangleright$  Ag :  $\lambda_1 = -1.\lambda_2 = -3.(sad)$

Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem

Principle Axis Theorei Nutshell



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Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination

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Test cases:

$$\blacktriangleright \mathbb{A}_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \mathbb{A}_2 = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}, \mathbb{A}_3 = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

Some minor struggling leads to::

- A<sub>4</sub> : λ<sub>4</sub> = +8, λ<sub>5</sub> = +1, (PDM, baop)
- $\blacktriangleright A_{0}: \lambda_{1} = \pm \sqrt{5} \lambda_{0} = -\sqrt{5} (\text{sad})$
- $|A_2|: \lambda_3 = -1.\lambda_2 = -3.(sad)$

Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination

Principle Axis Theorem

Nutshell



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- $A_1: \lambda_1 = +3, \lambda_2 = +1, (PDM, happy)$
- $\blacktriangleright \mathbb{A}_2: \lambda_1 = \pm \sqrt{5}, \lambda_2 = -\sqrt{5}, \text{ (sad)}$
- $A_3: \lambda_1 = -1, \lambda_2 = -3$ , (sad)

Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is...

Completing the square ⇔ Gaussian elimination

Principle Axis Theorem

Nutshell



- We recall with alacrity the totally amazing fact that real symmetric matrices always have (1) real eigenvalues, and (2) orthogonal eigenvectors forming a basis for R<sup>n</sup>.
- We now see that knowing the signs of the λs is also important...

Test cases:

$$\blacktriangleright \mathbb{A}_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \mathbb{A}_2 = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}, \mathbb{A}_3 = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

Some minor struggling leads to::

- $\mathbb{A}_1 : \lambda_1 = +3, \lambda_2 = +1,$  (PDM, happy),
- $\blacktriangleright \mathbb{A}_2 : \lambda_1 = \pm \sqrt{5}, \lambda_2 = -\sqrt{5}, \text{ (sad)}$
- $A_3: \lambda_1 = -1, \lambda_2 = -3, \text{ (sad)}$

Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is...

Completing the square  $\Leftrightarrow$ Gaussian elimination

Principle Axis Theorem

Nutshell



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• 
$$\mathbb{A}_1$$
 :  $\lambda_1 = +3, \lambda_2 = +1$ , (PDM, happy)

• 
$$\mathbb{A}_2: \lambda_1 = +\sqrt{5}, \lambda_2 = -\sqrt{5}, \text{ (sad)}$$

• 
$$\mathbb{A}_3 : \lambda_1 = -1, \lambda_2 = -3$$
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Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination

Principle Axis Theorem

Nutshell



Extremely Sneaky Result #632: If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A is real, then If A = A <sup>T</sup> and A = A <sup>T</sup> and A = A <sup></sup>	P	U	r	e	n	na	ac	dr	пe	s	S	•													Posi Mati				
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## Extremely Sneaky Result #632: If $A = A^T$ and A is real, then

- # +ve eigenvalues = # +ve pivots
- # -ve eigenvalues = # -ve pivots
- # 0 eigenvalues = # 0 pivots

#### Notes:

- Ereviously, we had for general 1. that
   [A] = 11.X = ± 11.0.
   The bonus here is for real symmetric 4.
   Eigenvalues are pivots come from very difference parts of finoar algobra.
  - Grázi/ pannedtien behvelete éigenvalues and pivot

#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

What a PDM is...

Identifying PDMs Completing the square  $\Leftrightarrow$ Gaussian elimination

Principle Axis Theorem

Nutshell



# Extremely Sneaky Result #632:

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#### Notes:



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### Notes:

- Previously, we had for general  $\mathbb{A}$  that  $|\mathbb{A}| = \prod \lambda_i = \pm \prod d_i$ .
  - The bonus here is for real symmetric A
- Eigenvalues are pivots come from very different parts of linear algebra.

Crazy connection between eigenvalues and pivots!

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Lecture 26

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Identifying PDMs Completing the square Gaussian elimination Principle Axis Theorem

Nutshell



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#### More notes:

- All very exciting: Pivots are much, much easier to compute.
- (cue balloons, streamers)

#### Check for our three examples:



Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

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#### Positive Definite Matrices (PDMs)

Lecture 26

What a PDM is...

Identifying PDMs Completing the square  $\Leftrightarrow$ Gaussian elimination

Gaussian elimination Principle Axis Theorem

Nutshell

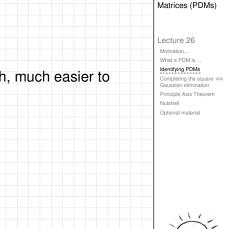


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Positive Definite

## Pivots and Eigenvalues:

#### More notes:

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### Check for our three examples:

• 
$$\mathbb{A}_1: d_1 = +2, d_2 = +\frac{3}{2}$$

- $\checkmark$  signs match with  $\lambda_1 = +3, \lambda_2 = +1$ .
- $\mathbb{A}_2: d_1 = \pm 2, d_2 = \pm \frac{5}{2}$ 
  - $\checkmark$  signs match with  $\lambda_1 = -\sqrt{5}, \lambda_2 = -\sqrt{5}.$
- $A_3: d_1 = -2, d_2 = -\frac{3}{2}$ 
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Positive Definite Matrices (PDMs)

Lecture 26

What a PDM is...

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- $\mathbb{A}_3$ :  $d_1 = -2$ ,  $d_2 = -\frac{3}{2}$ 
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#### Positive Definite Matrices (PDMs)

Lecture 26

What a PDM is...

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#### Positive Definite Matrices (PDMs)

Lecture 26

What a PDM is...

Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination

Principle Axis Theorem

Nutshell



 Let's show how the signs of eigenvalues match signs of pivots for

$$\mathbb{A}_2 = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}, \quad \lambda_{1,2} = \pm \sqrt{5}$$

Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

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 Let's show how the signs of eigenvalues match signs of pivots for

$$\mathbb{A}_2 = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}, \quad \lambda_{1,2} = \pm \sqrt{5}$$

Compute LU decomposition:

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & -\frac{5}{2} \end{bmatrix} = \mathbb{LU}$$

A<sub>2</sub> is symmetric, so we can go further:

$$A_{2} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -\frac{1}{2} & 1 & 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} = LDL^{T}$$

Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

What a PDM is... Identifving PDMs

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 Let's show how the signs of eigenvalues match signs of pivots for

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Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

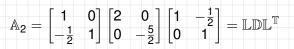
What a PDM is...

Identifying PDMs Completing the square Gaussian elimination Principle Axis Theorem

Nutshell



We're here:



- Now think about this matrix
  - $\mathbb{B}(\ell_{21}) = \begin{bmatrix} 1 & 0 \\ \ell_{21} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & \ell_{21} \\ 0 & 1 \end{bmatrix}$
- When  $\ell_{21} = -\frac{1}{2}$ , we have  $B(-\frac{1}{2}) = \mathbb{A}_2$ .

► Think about what happens as l<sub>21</sub> changes smoothly from - <sup>1</sup>/<sub>2</sub> to 0.



- Lecture 26 Motivation...
- What a PDM is...
- Identifying PDMs Completing the square Gaussian elimination Principle Axis Theorem
- Nutshell
  - Optional material



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$$\mathbb{B}(\ell_{21}) = \begin{bmatrix} 1 & 0 \\ \ell_{21} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & \ell_{21} \\ 0 & 1 \end{bmatrix}$$

• When  $\ell_{21} = -\frac{1}{2}$ , we have  $B(-\frac{1}{2}) = A_2$ .

Think about what happens as  $\ell_{21}$  changes smoothly from  $-\frac{1}{2}$  to 0.

Positive Definite Matrices (PDMs)

Lecture 26 Motivation...

What a PDM is...

Identifying PDMs Completing the square Gaussian elimination Principle Axis Theorem Nutshell



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$$\mathbb{A}_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} = \mathbb{L}\mathbb{D}\mathbb{L}^{\mathbb{T}}$$

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$$\mathbb{B}(\ell_{21}) = \begin{bmatrix} 1 & 0 \\ \ell_{21} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & \ell_{21} \\ 0 & 1 \end{bmatrix}$$

- When  $\ell_{21} = -\frac{1}{2}$ , we have  $B(-\frac{1}{2}) = \mathbb{A}_2$ .
- Think about what happens as ℓ<sub>21</sub> changes smoothly from -1/2 to 0.



#### Positive Definite Matrices (PDMs)

- Lecture 26 Motivation...
- What a PDM is...
- Identifying PDMs Completing the square Gaussian elimination Principle Axis Theorem Nutshell
- Optional material



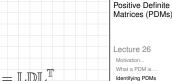
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$$\mathbb{B}(\ell_{21}) = \begin{bmatrix} 1 & 0 \\ \ell_{21} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & \ell_{21} \\ 0 & 1 \end{bmatrix}$$

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Completing the square Gaussian elimination Principle Axis Theorem Nutshell



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$$\mathbb{B}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{IDI} = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix}$$

Positive Definite Matrices (PDMs)

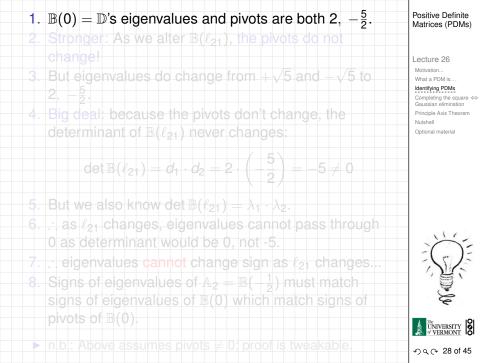
Lecture 26 Motivation... What a PDM is...

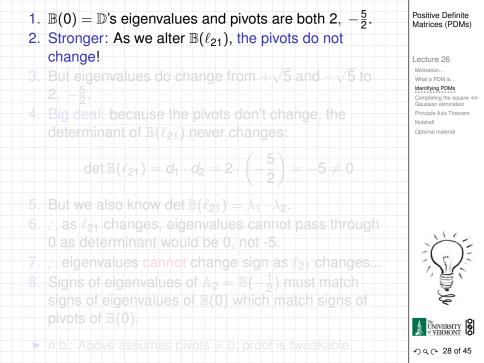
Identifying PDMs Completing the square <

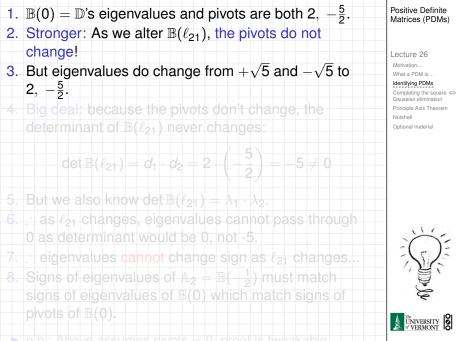
Principle Axis Theorem

Nutshell









 $\triangleright$  n.b.: Above assumes pivots  $\neq$  0; proof is tweakable

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- B(0) = D's eigenvalues and pivots are both 2, -<sup>5</sup>/<sub>2</sub>.
   Stronger: As we alter B(ℓ<sub>21</sub>), the pivots do not change!
- 3. But eigenvalues do change from  $+\sqrt{5}$  and  $-\sqrt{5}$  to 2,  $-\frac{5}{2}$ .
- 4. Big deal: because the pivots don't change, the determinant of  $\mathbb{B}(\ell_{21})$  never changes:

$$\det \mathbb{B}(\ell_{21}) = d_1 \cdot d_2 = 2 \cdot \left(-\frac{5}{2}\right) = -5 \neq 0$$

- 5. But we also know det  $\mathbb{B}(\ell_{21}) = \lambda_1 \cdot \lambda_2$
- 6. : as  $\ell_{21}$  changes, eigenvalues cannot pass through 0 as determinant would be 0, not -5.
- 7. : eigenvalues cannot change sign as  $\ell_{21}$  changes.
- Signs of eigenvalues of A<sub>2</sub> = B(-<sup>1</sup>/<sub>2</sub>) must match signs of eigenvalues of B(0) which match signs of pivots of B(0).

In.b.: Above assumes pivots ⊭ 0; proof is tweakable

Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem



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- 8. Signs of eigenvalues of  $\mathbb{A}_2 = \mathbb{B}(-\frac{1}{2})$  must match signs of eigenvalues of  $\mathbb{B}(0)$  which match signs of pivots of  $\mathbb{B}(0)$ .
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Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem





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   Stronger: As we alter B(ℓ<sub>21</sub>), the pivots do not change!
   Dut since real uses the shares form to √E and w/E to all a shares form to all a
- 3. But eigenvalues do change from  $+\sqrt{5}$  and  $-\sqrt{5}$  to 2,  $-\frac{5}{2}$ .
- 4. Big deal: because the pivots don't change, the determinant of  $\mathbb{B}(\ell_{21})$  never changes:

det 
$$\mathbb{B}(\ell_{21}) = d_1 \cdot d_2 = 2 \cdot \left(-\frac{5}{2}\right) = -5 \neq 0$$

/ **F** 

- 5. But we also know det  $\mathbb{B}(\ell_{21}) = \lambda_1 \cdot \lambda_2$ .
- ∴ as l<sub>21</sub> changes, eigenvalues cannot pass through 0 as determinant would be 0, not -5.
- 7. ∴ eigenvalues cannot change sign as l<sub>21</sub> changes.
   8. Signs of eigenvalues of A<sub>2</sub> = B(-<sup>1</sup>/<sub>2</sub>) must match signs of eigenvalues of B(0) which match signs of pivots of B(0).

In.b.: Above assumes pivots ⊭ 0; proof is tweakable

Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs

Completing the square Gaussian elimination Principle Axis Theorem





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#### Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is...

Identifying PDMs Completing the square Gaussian elimination Principle Axis Theorem



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Lecture 26 Motivation... What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem

Nutshell Optional material



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Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is...

Identifying PDMs Completing the square Gaussian elimination Principle Axis Theorem Nutshell



#### Can see argument extends to n by n's.

- Take  $\mathbb{A} = \mathbb{A}^{T} = \mathbb{L}\mathbb{D}\mathbb{L}^{T}$  and smoothly change  $\mathbb{L}$  to
- Vite  $\hat{\mathbb{L}}(t) = \mathbb{I} + t(\mathbb{L} \mathbb{I})$  and
  - $\mathbb{B}(t) = \hat{\mathbb{L}}(t) \mathbb{D} \, \hat{\mathbb{L}}(t)^{\mathrm{T}}$
- When t = 1, we have  $\hat{\mathbb{L}}(1) = \mathbb{L}$  and  $\mathbb{B}(1) = \mathbb{A}$ .
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- Again, pivots don't change as we move t from 1 to 0, and determinant must stay the same.
- Same story: eigenvalues cannot cross zero and must have the same signs for all t, including t = 0 when eigenvalues and pivots are equal A = D.

#### Positive Definite Matrices (PDMs)

Lecture 26

What a PDM is

Identifying PDMs

Completing the square  $\Leftrightarrow$  Gaussian elimination

Principle Axis Theorem

Nutshell



- Can see argument extends to n by n's.
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Positive Definite Matrices (PDMs)

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What a PDM is...

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Principle Axis Theorem



# Outline

Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

What a PDM is...

Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem

Principle Axis Theor

Nutshell

Optional material

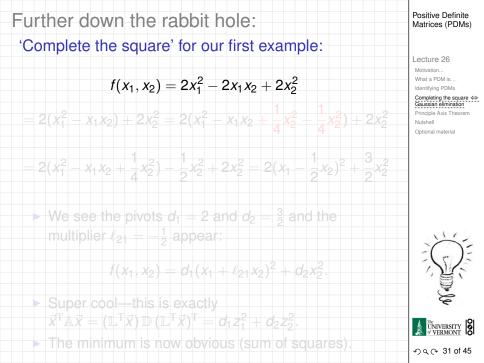
# Lecture 26

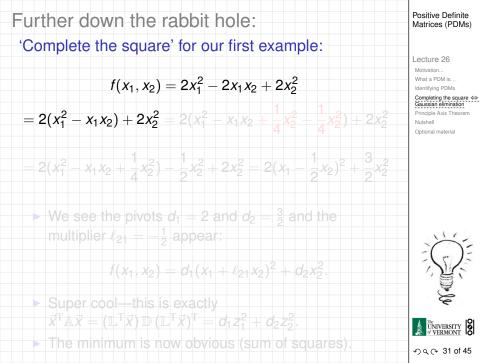
Motivation... What a PDM is... Identifying PDMs

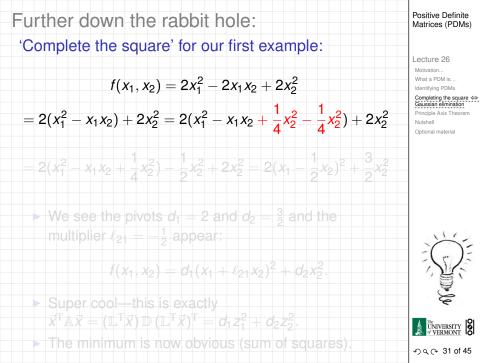
#### Completing the square ⇔ Gaussian elimination

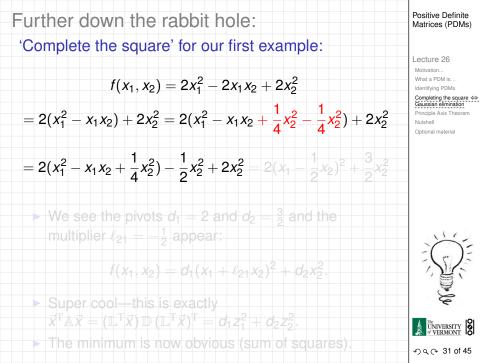
Principle Axis Theorem Nutshell Optional material











# Further down the rabbit hole:

#### 'Complete the square' for our first example:

$$f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 2x_2^2$$

$$= 2(x_1^2 - x_1x_2) + 2x_2^2 = 2(x_1^2 - x_1x_2 + \frac{1}{4}x_2^2 - \frac{1}{4}x_2^2) + 2x_2^2$$

$$=2(x_1^2-x_1x_2+\frac{1}{4}x_2^2)-\frac{1}{2}x_2^2+2x_2^2=2(x_1-\frac{1}{2}x_2)^2+\frac{3}{2}x_2^2$$

We see the pivots  $d_1 = 2$  and  $d_2 = \frac{3}{2}$  and the multiplier  $\ell_{21} = -\frac{1}{2}$  appear:



- Super cool—this is exactly  $\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x} = (\mathbb{L}^{\mathrm{T}} \vec{x}) \mathbb{D} (\mathbb{L}^{\mathrm{T}} \vec{x})^{\mathrm{T}} = d_1 z_1^2 + d_2 z_2^2.$
- The minimum is now obvious (sum of squares)

#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

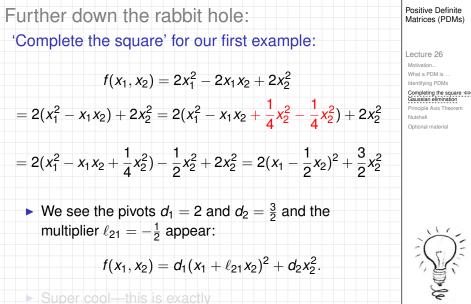
What a PDM is...

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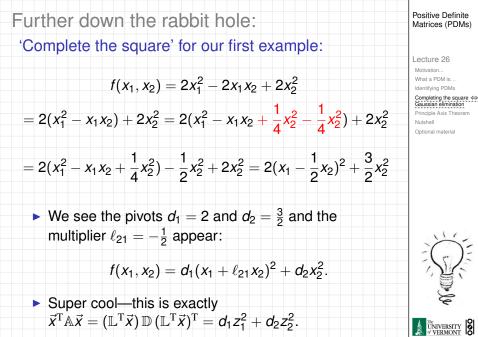




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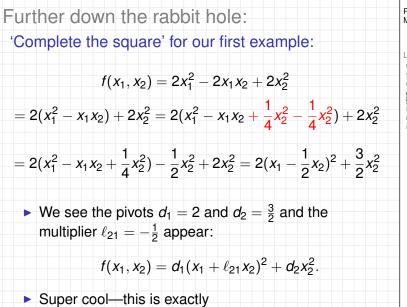
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 $\vec{x}^{\mathrm{T}} \mathbb{A} \vec{x} = (\mathbb{L}^{\mathrm{T}} \vec{x}) \mathbb{D} (\mathbb{L}^{\mathrm{T}} \vec{x})^{\mathrm{T}} = d_1 z_1^2 + d_2 z_2^2.$ 

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Positive Definite Matrices (PDMs)

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#### Another example:

Take the matrix A<sub>2</sub>:

$$\mathbf{x}(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Complete the square:

- $f(x_1, x_2) = 2x_1^2 2x_1x_2 2x_2^2 = 2(x_1 \frac{1}{2}x_2)^2 \frac{3}{2}x_2^2$
- Matches: Pivots  $d_1 = 2$ ,  $d_2 = -\frac{5}{2}$ , so  $x_1 = x_2 = 0$  is a saddle.

Completing the square matches up with elimination...

# Positive Definite Matrices (PDMs)

Lecture 26

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- Matches: Pivots d<sub>1</sub> = 2, d<sub>2</sub> = −<sup>5</sup>/<sub>2</sub>, so x<sub>1</sub> = x<sub>2</sub> = 0 is a saddle.
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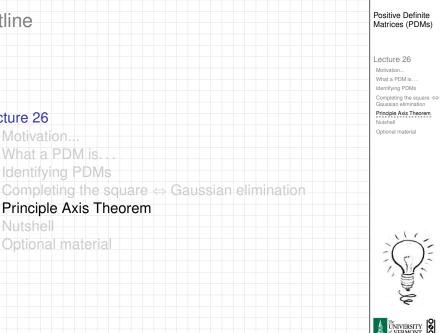


### Outline

Lecture 26

Motivation...

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### Back to our second simple problem:

• Graph  $2x_1^2 + 2x_1x_2 + 2x_2^2 = 1$ .

- We'll simplify with linear algebra to find an equation of an ellipse...
- From before, our equation can be rewritten as



Again use spectral decomposition,  $\mathbb{A} = \mathbb{Q} \wedge \mathbb{Q}^{\mathrm{T}}$ , to diagonalize giving  $(\mathbb{Q}^{\mathrm{T}} \vec{x})^{\mathrm{T}} \wedge (\mathbb{Q}^{\mathrm{T}} \vec{x}) = 1$  where

#### Positive Definite Matrices (PDMs)

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Principle Axis Theorem Nutshell



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#### Positive Definite Matrices (PDMs)

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Positive Definite Matrices (PDMs)

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► Again use spectral decomposition, A = Q ∧ Q<sup>T</sup>, to diagonalize giving (Q<sup>T</sup> x)<sup>T</sup> ∧ (Q<sup>T</sup> x) = 1 where

$$\mathbb{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\mathbb{Q}} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{\Lambda} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{\mathbb{Q}^{\mathrm{T}}}$$

#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

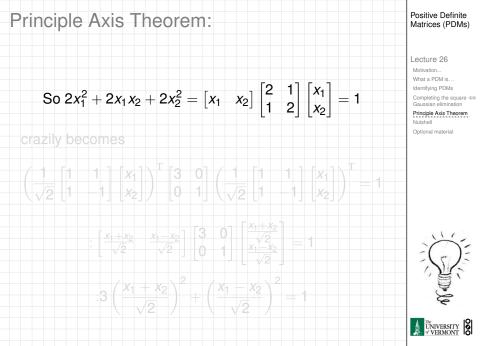
What a PDM is...

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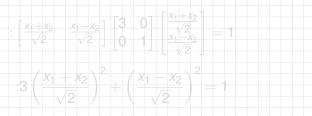


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So 
$$2x_1^2 + 2x_1x_2 + 2x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

crazily becomes

$$\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)^{\mathrm{T}} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)^{\mathrm{T}} = 1$$



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$$\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix}3&0\\0&1\end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)^{\mathrm{T}}=1$$

$$\begin{bmatrix} \underline{x_1+x_2} & \underline{x_1-x_2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{x_1+x_2} \\ \sqrt{2} \\ \underline{x_1-x_2} \\ \sqrt{2} \end{bmatrix} = \cdot$$

Lecture 26

Motivation...

What a PDM is...

Identifying PDMs

Completing the square  $\Leftrightarrow$  Gaussian elimination

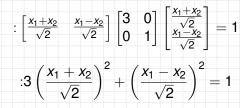
Principle Axis Theorem Nutshell



So 
$$2x_1^2 + 2x_1x_2 + 2x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

crazily becomes

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#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

What a PDM is...

Identifying PDMs

Completing the square Gaussian elimination

Principle Axis Theorem Nutshell



If we change to eigenvector coordinate system,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbb{Q}^{\mathrm{T}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1 + x_2}{\sqrt{2}} \\ \frac{x_1 - x_2}{\sqrt{2}} \end{bmatrix}$$

•

1.

then our equation simplifies greatly:

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} =$$

which is just

$$3 \cdot u_1^2 + 1 \cdot u_2^2 = 1$$

Very nice! PDM : ellipse.

#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation...

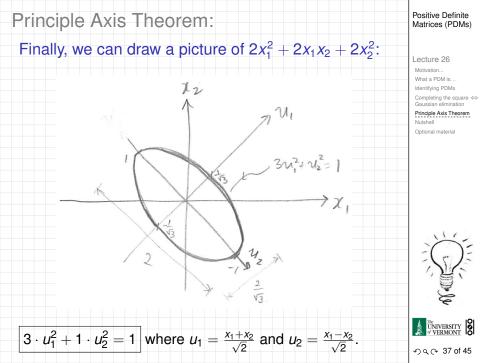
What a PDM is...

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### Outline

Positive Definite Matrices (PDMs) Lecture 26 Motivation What a PDM is Identifying PDMs Completing the square  $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem Lecture 26 Nutshell Optional material Motivation... Identifying PDMs Principle Axis Theorem Nutshell



### • $\vec{x}^{T} \mathbb{A} \vec{x}$ is a commonly occurring construction.

- Big deals: Positive Definiteness and Semi-Positive Definiteness of A.
- Positive eigenvalues : PDM.
- Non-negative eigenvalues : SPDM.
- Signs of pivots (easy test) match signs of eigenvalues.
- Gaussian elimination  $\equiv$  completing the square.
- Standard questions: determine if a matrix is a PDM, convert a quadratic function into matrix x<sup>T</sup>Ax, sketch a quadratic curve (e.g., an ellipse).

#### Positive Definite Matrices (PDMs)

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Principle Axis Theorem



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#### Positive Definite Matrices (PDMs)

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#### Positive Definite Outline Matrices (PDMs) Lecture 26 Motivation What a PDM is Identifying PDMs Completing the square $\Leftrightarrow$ Gaussian elimination Principle Axis Theorem Lecture 26 Nutshell Optional material Motivation... Identifying PDMs Principle Axis Theorem Nutshell **Optional material**



#### Positive Definite Matrices (PDMs)

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# For a real symmetric $\mathbb{A}$ , if all upper left determinants of $\mathbb{A}$ are +ve, so are $\mathbb{A}$ 's eigenvalues, and vice versa.



ST #731:

#### Positive Definite Matrices (PDMs)

Lecture 26

Motivation... What a PDM is

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Completing the square  $\Leftrightarrow$ Gaussian elimination

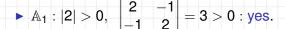
Principle Axis Theorem

Nutshell

Optional material

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ST #731:



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#### Positive Definite Matrices (PDMs)

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## Lecture 26 Motivation...

Identifying PDMs

Completing the square  $\Leftrightarrow$ Gaussian elimination

Principle Axis Theorem

Nutshell

Optional material

### Check:

ST #731:

• 
$$\mathbb{A}_1: |2| > 0, \quad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0:$$
 yes.

For a real symmetric  $\mathbb{A}$ , if all upper left

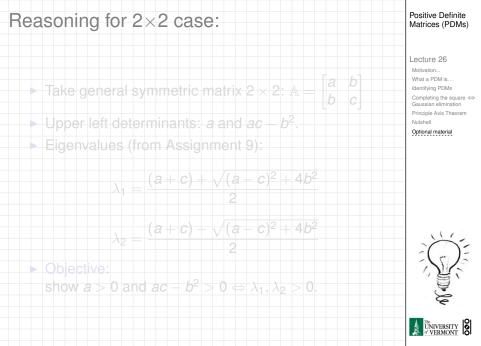
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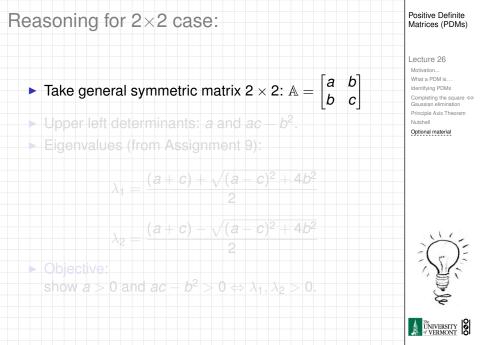
• 
$$\mathbb{A}_2: |2| > 0, \quad \begin{vmatrix} 2 & -1 \\ -1 & -2 \end{vmatrix} = -5 < 0: \text{no.}$$

• 
$$\mathbb{A}_3: |-2| < 0, \quad \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = 3 > 0: no.$$





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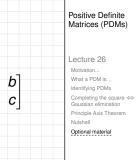


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- Take general symmetric matrix  $2 \times 2$ :  $\mathbb{A} = \begin{vmatrix} a & b \\ b & c \end{vmatrix}$
- Upper left determinants: a and  $ac b^2$ .
- Eigenvalues (from Assignment 9):



Objective: show a > 0 and  $ac = b^2 > 0 \Leftrightarrow \lambda_1 \lambda_2 > 0$ 





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$$\lambda_1 = rac{(a+c) + \sqrt{(a-c)^2 + 4b^2}}{2}$$
 $\lambda_2 = rac{(a+c) - \sqrt{(a-c)^2 + 4b^2}}{2}$ 

Objective:

show a > 0 and  $ac - b^2 > 0 \Leftrightarrow \lambda_1, \lambda_2 > 0$ .

#### Positive Definite Matrices (PDMs)



Principle Axis Theorem

Nutshell



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• Objective:  
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$$a > 0$$
 and  $ac - b^2 > 0 \Leftrightarrow \lambda_1, \lambda_2 > 0$ 

#### Positive Definite Matrices (PDMs)

Lecture 26 Motivation...

What a PDM is...

Identifying PDMs

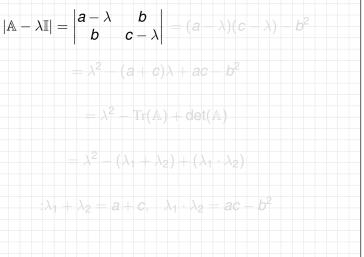
Completing the square  $\Leftrightarrow$  Gaussian elimination

Principle Axis Theorem

Nutshell



### Reuse previous sneakiness:



Positive Definite Matrices (PDMs)

Lecture 26

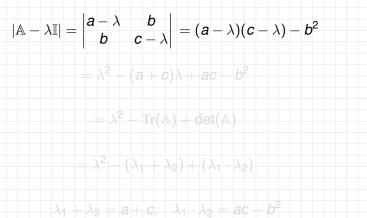
Motivation... What a PDM is... Identifying PDMs Completing the square  $\Leftrightarrow$ Gaussian elimination

Principle Axis Theorem

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### Reuse previous sneakiness:



#### Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs Completing the square ⇔ Gaussian elimination

Principle Axis Theorem

Nutshell



### Reuse previous sneakiness:

$$|\mathbb{A}-\lambda\mathbb{I}|=egin{bmatrix} oldsymbol{a}-\lambda&oldsymbol{b}\ oldsymbol{b}&oldsymbol{c}-\lambda \end{bmatrix}=(oldsymbol{a}-\lambda)(oldsymbol{c}-\lambda)-oldsymbol{b}^2$$

$$= \lambda^2 - (a+c)\lambda + ac - b^2$$

$$= \lambda^2 - (\lambda_1 + \lambda_2) + (\lambda_1 \cdot \lambda_2)$$
$$\lambda_1 + \lambda_2 = a + c, \quad \lambda_1 \cdot \lambda_2 = a c - b^2$$

#### Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs

Completing the square  $\Leftrightarrow$  Gaussian elimination

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Nutshell



### Reuse previous sneakiness:

$$|\mathbb{A} - \lambda \mathbb{I}| = \begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = (a - \lambda)(c - \lambda) - b^2$$

$$=\lambda^2-(a+c)\lambda+ac-b^2$$

$$= \lambda^2 - \operatorname{Tr}(\mathbb{A}) + \operatorname{det}(\mathbb{A})$$



#### Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs Completing the square ⇔ Gaussian elimination Principle Axis Theorem

Nutshell



# Reasoning for $2 \times 2$ case:

# Reuse previous sneakiness:

$$|\mathbb{A} - \lambda \mathbb{I}| = egin{pmatrix} a - \lambda & b \ b & c - \lambda \end{bmatrix} = (a - \lambda)(c - \lambda) - b^2$$

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$$\lambda^2-(\lambda_1+\lambda_2)+(\lambda_1\cdot\lambda_2)$$

$$\lambda_1 + \lambda_2 = a + c, \quad \lambda_1 \cdot \lambda_2 = ac - b^2$$

#### Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs Completing the square ⇔ Gaussian elimination

Principle Axis Theorem

Nutshell



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#### Positive Definite Matrices (PDMs)

Lecture 26 Motivation... What a PDM is... Identifying PDMs Completing the square ⇔

Gaussian elimination

Principle Axis Theorem

Nutshell



Show <i>a</i> > 0,	$ac-b^2$	$>$ 0 $\Leftrightarrow \lambda_1, \ \lambda_2 >$ 0:	Positive Definite Matrices (PDMs)
Show ":":			Lecture 26
Given en			What a PDM is Identifying PDMs
			Completing the square Gaussian elimination
elgenvalues			Principle Axis Theorem Nutshell
► Given a > 0			Optional material
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are positive			
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#### Positive Definite Matrices (PDMs)

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Optional material



# Show ":": Given ac + b<sup>2</sup> > 0 then λ<sub>1</sub> · λ<sub>2</sub> > 0, so both eigenvalues are positive or both are negative. Given a > 0 then c > 0 b/c otherwise ac + b<sup>2</sup> < 0.</li> This means a + c = λ<sub>1</sub> + λ<sub>2</sub> > 0 → both eigenvalues are positive.

### Show " $\Leftarrow$ ":

Given N., X<sub>2</sub> > 0, then ac 4<sup>2</sup> - X<sub>1</sub> - X<sub>2</sub> > 0
 Kopw a + c - X<sub>1</sub> + X<sub>2</sub> > 0, so either a - c > 0, or one is nepative.
 But again, ac - a<sup>2</sup> > 0 implies a, c must have same close a - c

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#### Positive Definite Matrices (PDMs)

## Show ":":

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- Given a > 0 then c > 0 b/c otherwise  $ac b^2 < 0$ .
- This means  $a + c = \lambda_1 + \lambda_2 > 0 \rightarrow$  both eigenvalues are positive.

## Show "⇐":

 $= \underline{Given V_{1}, V_{2} > 0, \text{ then } ac = \underline{A^{2}} = \underline{V_{1} + \underline{V_{2} > 0}}$   $= \underline{Kop}_{4} = \underline{c} = \underline{V_{1} + \underline{V_{2} > 0}}, \text{ so either } \underline{a} = \underline{c} > 0, \text{ or ob}$   $= \underline{Is nepative.}$   $= \underline{But again, ac = \underline{A^{2} > 0 implies a, conuscitave same}$ 

Lecture 26 Motivation... What a PDM is... Identifying PDMs Completing the square ⇔ Gaussian elimination Principle Axis Theorem Nutshell Optional material



#### Positive Definite Matrices (PDMs)

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#### Positive Definite Matrices (PDMs)

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#### Positive Definite Matrices (PDMs)

## Show ":":

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## Upshot: We can compute determinants instead of eigenvalues to find signs.

- But: Computing determinants still isn't a picnic either...
- A much better way is to use the connection between pivots and eigenvalues.
- Another weird connection.

Positive Definite Matrices (PDMs)

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