Chapter 3/4: Lecture 15

Linear Algebra MATH 124, Fall, 2010

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Ch 3/4: Lec 15

Review for Exam 2

Words



Outline Review for Exam 2 Words Pictures Review for Exam 2





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Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1–4.3)
- Main pieces:

As always, want 'doing' and 'understanding' and abilities.



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- ▶ Chapter 3 and Chapter 4 (Sections 4.1–4.3)
- ► Main pieces:
 - 1. Big Picture of $A\vec{x} = b$
 - 2. Projections and the normal equation
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- Main pieces:
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- ► Chapter 3 and Chapter 4 (Sections 4.1–4.3)
- Main pieces:
 - 1. Big Picture of $A\vec{x} = \vec{b}$ Must be able to draw the big picture!
 - 2. Projections and the normal equation
- As always, want 'doing' and 'understanding' and abilities.





- Vector space concept and definition.

- Various techniques for finding bases and orthogonal





- Vector space concept and definition.
- Subspace definition (three conditions).
- Concept of a spanning set of vectors
- Concept of a basis
- Basis = minimal spanning set
- Concept of orthogonal complement
- Various techniques for finding bases and orthogonal complements.





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Fundamental Theorem of Linear Algebra:

- Applies to any $m \times n$ matrix A.
- Symmetry of A and A^T.
- Column space C(A) ⊂ R^m.
- ► Left Nullspace $N(A^{T}) \subset R^{m}$.
- $| \dim C(A) + \dim N(A^T) | = r + (m r) = m$
- Orthogonality: $C(A) \otimes N(A^T) = R^m$
- Row space $C(A^{T}) \subset R^{n}$.
- (Right) Nullspace $N(A) \subset R^n$.
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Fundamental Theorem of Linear Algebra:

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- dim C(A) + dim $N(A^{T}) = r + (m + r) = m$
- ightharpoonup Orthogonality: $C(A) \otimes N(A^T) = R^m$
- ▶ Row space $C(A^T) \subset R^n$.
- (Right) Nullspace $N(A) \subset R^n$.
- $\int dim C(A^{T}) + dim N(A) = r + (n r) = n$
- Orthogonality: $C(A^T) \otimes N(A) = B^T$



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Fundamental Theorem of Linear Algebra:

- ▶ Applies to any $m \times n$ matrix A.
- Symmetry of A and A^T.
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- (Right) Nullspace $N(A) \subset \mathbb{R}^n$.
- $| \dim C(A^T) + \dim M(A) | = r + (n r) + n$
- Orthogonality: C(A^T) ⊗ N(A) + Rⁿ

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Fundamental Theorem of Linear Algebra:

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- Enough to find bases for subspaces
- ► Be able to reduce A to R
- Identify pivot columns and free columns
- Rank r of A = # pivot columns.
- Know that relationship between R's columns hold for A's columns.
- ► Warning: R's columns do not give a basis for C(A)
- ▶ But find pivot columns in R, and same columns in A





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- Reduce $[A \mid \vec{b}]$ where \vec{b} is general.
- Find conditions on b's elements for a solution to $A\vec{x} = \vec{b}$ to exist
- ► Basis for row space = non-zero rows in R (easy!)
- ► Alternate basis for column space = non-zero rows in reduced form of A^T (easy!)





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- ▶ Reduce $[A | \vec{b}]$ where \vec{b} is general.
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More on bases for column and row space:

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- Basis for nullspace obtained by solving $A\vec{x} = 0$
- Always express pivot variables in terms of free variables.
- Free variables are unconstrained (can be any real number)
- # free variables = n # pivot variables = n r = dim N(A).
- Similarly find basis for $N(A^T)$ by solving $A^T\vec{y} = \vec{0}$.



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- Number of solutions to $A\vec{x} = \vec{b}$:
 - 1. If $b \notin C(A)$, there are no solutions
 - 2. If $b \in C(A)$ there is either one unique solution of infinitely many solutions.





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- 1. If $\vec{b} \notin C(A)$, there are no solutions.
- 2. If $b \in C(A)$ there is either one unique solution or infinitely many solutions.



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- 1. If $\vec{b} \notin C(A)$, there are no solutions.
- 2. If $\vec{b} \in C(A)$ there is either one unique solution or infinitely many solutions.
 - Number of solutions now depends entirely on N(
 - If dim N(A) = n r > 0, then there are infinitely many solutions.
 - If dim N(A) = n r = 0, then there is one solution.



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- Understand how to project a vector \vec{b} onto a line in direction of \vec{a} .
- $b = \vec{p} + \vec{e}$
- \vec{p} = that part of \vec{b} that lies in the line:

$$\vec{p} = \vec{a}^{\mathrm{T}} \vec{b} \vec{a} \left(= \vec{a} \vec{a}^{\mathrm{T}} \vec{b} \right)$$

- \vec{e} = that part of \vec{b} that is orthogonal to the line.
- Understand generalization to projection onto subspaces.
- Understand construction and use of subspace projection operator *P*:

$$\vec{P} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}},$$

where A's columns form a subspace basis.



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- Understand how to project a vector \vec{b} onto a line in direction of \vec{a} .
- b = p + e
- \vec{p} = that part of b that lies in the line:

$$\vec{p} = \vec{a}^{T} \vec{b} \vec{a} \left(\begin{array}{c} \vec{a} \vec{a}^{T} \\ \vec{a}^{T} \vec{a} \end{array} \right)$$

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- Understand how to project a vector \vec{b} onto a line in direction of \vec{a} .
- $ightharpoonup ec{b} = ec{p} + ec{e}$
- \vec{p} = that part of *b* that lies in the line:

- $\vec{e} = \text{that part of } \vec{b} \text{ that is orthogonal to the line.}$
- Understand generalization to projection onto subspaces.
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where A's columns form a subspace basis.

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- Understand how to project a vector \vec{b} onto a line in direction of \vec{a} .
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- \vec{e} = that part of \vec{b} that is orthogonal to the line.
- Understand generalization to projection onto subspaces.
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$$\vec{P} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}},$$

where A's columns form a subspace basis.



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Normal equation for $A\vec{x} = \vec{b}$:

If $\vec{b} \notin C(A)$, project \vec{b} onto C(A).

Nhow $\vec{p} \in C(A)$ so $\exists \vec{x}_{*}$ such that $A\vec{x}_{*} + \vec{p}$.

- Write projection of \vec{b} as \vec{p} .
- Error vector must be orthogonal to column space so
- $A^{\mathrm{T}}\vec{e} = A^{\mathrm{T}}(\vec{b} \vec{p}) = \vec{0}$
- ► Rearrange:

Since $A\vec{x}_* = \vec{p}$, we end up with

$$A^{\mathrm{T}}A\vec{x}_{*}=A^{\mathrm{T}}\vec{b}.$$

This is linear algebra's normal equation;





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Normal equation for $A\vec{x} = \vec{b}$:

- ▶ If $\vec{b} \notin C(A)$, project \vec{b} onto C(A).
- Write projection of b as p.
- Nnow $\vec{p} \in C(A)$ so $\exists \vec{x}_*$ such that $A\vec{x}_* = \vec{p}$.
- Error vector must be orthogonal to column space so $A^T \vec{e} = A^T (\vec{b} \vec{\rho}) = \vec{0}$.
- Rearrange:

Since $A\vec{x}_* = \vec{p}$, we end up with

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- Normal equation for $A\vec{x} = \vec{b}$:
 - If $\vec{b} \notin C(A)$, project \vec{b} onto C(A).
 - Write projection of \vec{b} as \vec{p} .
 - Now $\vec{p} \in C(A)$ so $\exists \vec{x}_*$ such that $A\vec{x}_* = \vec{p}$.
 - Error vector must be orthogonal to column space so $A^{T} \vec{e} = A^{T} (\vec{b} \vec{p}) = \vec{0}$.
 - Rearrange:

Since $A\vec{x}_* = \vec{p}$, we end up with

$$A^{\mathrm{T}}A\vec{x}_{*}=A^{\mathrm{T}}\vec{b}$$

This is linear algebra's normal equation;

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Normal equation for $A\vec{x} = \vec{b}$:

- ▶ If $\vec{b} \notin C(A)$, project \vec{b} onto C(A).
- ▶ Write projection of \vec{b} as \vec{p} .
- ► Know $\vec{p} \in C(A)$ so $\exists \vec{x}_*$ such that $A\vec{x}_* = \vec{p}$.
- Error vector must be orthogonal to column space so $A^{T} \vec{e} = A^{T} (\vec{b} \vec{p}) = \vec{0}$.
- Rearrange:

$$A^{\mathrm{T}}\vec{p} = A^{\mathrm{T}}\vec{b}$$

Since $A\vec{x}_* = \vec{p}$, we end up with

$$A^{\mathrm{T}}A\vec{x}_{*}=A^{\mathrm{T}}\vec{b}$$

This is linear algebra's normal equation; \vec{x} is our best solution to $A\vec{x} = \vec{b}$



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Normal equation for $A\vec{x} = \vec{b}$:

- ▶ If $\vec{b} \notin C(A)$, project \vec{b} onto C(A).
- Write projection of \vec{b} as \vec{p} .
- ► Know $\vec{p} \in C(A)$ so $\exists \vec{x}_*$ such that $A\vec{x}_* = \vec{p}$.
- From vector must be orthogonal to column space so $A^T \vec{e} = A^T (\vec{b} \vec{p}) = \vec{0}$.
- Rearrange

$$A^{\mathrm{T}}\vec{p} = A^{\mathrm{T}}\vec{b}$$

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- From vector must be orthogonal to column space so $A^T \vec{e} = A^T (\vec{b} \vec{p}) = \vec{0}$.
- Rearrange:

$$\mathbf{A}^{\mathrm{T}}\vec{\mathbf{p}}=\mathbf{A}^{\mathrm{T}}\vec{\mathbf{b}}$$

Since $A\vec{x}_* = \vec{\rho}$, we end up with

$$A^{\mathrm{T}}A\vec{x}_{*}=A^{\mathrm{T}}\vec{b}$$

This is linear algebra's normal equation;
x̄_k is our best solution to Ax̄ = b̄.



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Normal equation for $A\vec{x} = \vec{b}$:

- ▶ If $\vec{b} \notin C(A)$, project \vec{b} onto C(A).
- ▶ Write projection of \vec{b} as \vec{p} .
- ► Know $\vec{p} \in C(A)$ so $\exists \vec{x}_*$ such that $A\vec{x}_* = \vec{p}$.
- From vector must be orthogonal to column space so $A^T \vec{e} = A^T (\vec{b} \vec{p}) = \vec{0}$.
- ► Rearrange:

$$A^{\mathrm{T}}\vec{p} = A^{\mathrm{T}}\vec{b}$$

Since $A\vec{x}_* = \vec{p}$, we end up with

$$A^{\mathrm{T}}A\vec{x}_*=A^{\mathrm{T}}\vec{b}.$$

This is linear algebra's normal equation;
x̄, is our best solution to Ax̄ = b̄.



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- ▶ If $\vec{b} \notin C(A)$, project \vec{b} onto C(A).
- ▶ Write projection of \vec{b} as \vec{p} .
- ► Know $\vec{p} \in C(A)$ so $\exists \vec{x}_*$ such that $A\vec{x}_* = \vec{p}$.
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- ► Rearrange:

$$A^{\mathrm{T}}\vec{p} = A^{\mathrm{T}}\vec{b}$$

Since $A\vec{x}_* = \vec{p}$, we end up with

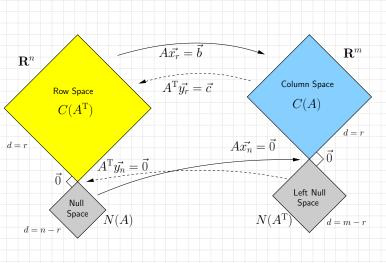
$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\vec{\mathbf{x}}_{*}=\mathbf{A}^{\mathrm{T}}\vec{\mathbf{b}}.$$

This is linear algebra's normal equation; \vec{x}_* is our best solution to $A\vec{x} = \vec{b}$.





The symmetry of $A\vec{x} = \vec{b}$ and $A^{T}\vec{y} = \vec{c}$:



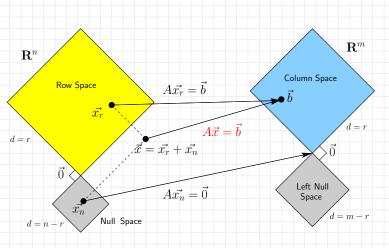
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How $A\vec{x} = \vec{b}$ works:



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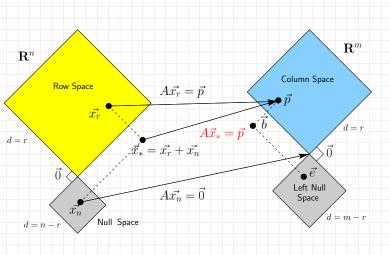
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Best solution \vec{x}_* when $\vec{b} = \vec{p} + \vec{e}$:



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The fourfold ways of $A\vec{x} = \vec{b}$:

case	example R	big picture	# solutions
m = r n = r		→	1 always
m = r, $n > r$			∞ always
m > r, n = r	1 0 0 1 0 0		0 or 1
m > r, n > r	1 0 %1 0 1 %2 0 0 0 0 0 0		0 or ∞

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