## Chapter 3/4: Lecture 15

## Linear Algebra

MATH 124, Fall, 2010

## Prof. Peter Dodds

Department of Mathematics \& Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont



UNIVERSITY VERMONT

## Outline

## Review for Exam 2

## Words

Pictures


The
UNIVERSITY of VERMONT
$\left|\begin{array}{l}0 \\ 0 \\ 0\end{array}\right|$ ๑ดく 2 of 16

## Basics:

Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1-4.3)


## Basics:

Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1-4.3)
- Main pieces:



## Basics:

Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1-4.3)
- Main pieces:

1. Big Picture of $A \vec{x}=\vec{b}$

- As always, want 'doing' and 'understanding' and abilities.


## Basics:

Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1-4.3)
- Main pieces:

1. Big Picture of $A \vec{x}=\vec{b}$
2. Projections and the normal equation

- As always, want 'doing' and 'understanding' and abilities.


## Basics:

Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1-4.3)
- Main pieces:

1. Big Picture of $A \vec{x}=\vec{b}$
2. Projections and the normal equation

- As always, want 'doing' and 'understanding' and abilities.


UNIVERSITY of VERMONT

## Basics:

Sections covered on second midterm:

- Chapter 3 and Chapter 4 (Sections 4.1-4.3)
- Main pieces:

1. Big Picture of $A \vec{x}=\vec{b}$

Must be able to draw the big picture!
2. Projections and the normal equation

- As always, want 'doing' and 'understanding' and abilities.


## Stuff to know/understand

## Vector Spaces:

- Vector space concept and definition.

Subspace definition (three conditions)
Concept of a spanning set of vectors
Concent of a basis
Basis = minimal spanning set.
Concept of orthogonal complement.

- Various techniques for finding bases and orthogonal complements.


Zhe UNIVERSITY of VERMONT

つの® 4 of 16

## Stuff to know／understand

## Vector Spaces：

－Vector space concept and definition．
－Subspace definition（three conditions）．
Concept of a spanning set of vectors．

Basis＝minimal spanning set．

Various techniques for finding bases and orthogonal complements．

## Words

Pictures


## Stuff to know/understand

## Vector Spaces:

- Vector space concept and definition.
- Subspace definition (three conditions).
- Concept of a spanning set of vectors.

Basis = minimal spanning set.

Various techniques for finding bases and orthogonal complements.


UNe of VERMONT

## Stuff to know/understand

## Vector Spaces:

- Vector space concept and definition.
- Subspace definition (three conditions).
- Concept of a spanning set of vectors.
- Concept of a basis.

Basis = minimal spanning set. Various techniques for finding bases and orthogonal complements.


The ${ }^{\text {Un }}$ NVERSITY of VERMONT

## Stuff to know/understand

## Vector Spaces:

- Vector space concept and definition.
- Subspace definition (three conditions).
- Concept of a spanning set of vectors.
- Concept of a basis.
- Basis = minimal spanning set.



## Stuff to know/understand

## Vector Spaces:

- Vector space concept and definition.
- Subspace definition (three conditions).
- Concept of a spanning set of vectors.
- Concept of a basis.
- Basis = minimal spanning set.
- Concept of orthogonal complement.
- Various techniques for finding bases and orthogonal complements.
$\left|\begin{array}{l}0 \\ 0 \\ 0\end{array}\right|$


## Stuff to know／understand

## Vector Spaces：

－Vector space concept and definition．
－Subspace definition（three conditions）．
－Concept of a spanning set of vectors．
－Concept of a basis．
－Basis＝minimal spanning set．
－Concept of orthogonal complement．
－Various techniques for finding bases and orthogonal complements．
$\left|\begin{array}{l}0 \\ 0\end{array}\right|$

## Stuff to know／understand：

Fundamental Theorem of Linear Algebra：
Applies to any $m \times n$ matrix $A$ ．
$\square$ Left Nullspace $N\left(A^{T}\right) \in R^{m}$ ． dimf $C(A)+\operatorname{dim} N\left(A^{T}\right)-r+(m-r)=m$ Orithogonality：$C(A) \otimes N\left(A^{T}\right)=8 m$

Row space （Right）Nullspace N（A） $R^{n}$


Review for Exam 2

Pictures


## Stuff to know/understand:

Fundamental Theorem of Linear Algebra:

- Applies to any $m \times n$ matrix $A$.
- Symmetry of $A$ and $A^{\mathrm{T}}$.

Row space (Right) Nullspace N(A) $R^{n}$


## Stuff to know/understand:

Fundamental Theorem of Linear Algebra:

- Applies to any $m \times n$ matrix $A$.
- Symmetry of $A$ and $A^{\mathrm{T}}$.
- Column space $C(A) \subset R^{m}$.
- Left Nullspace $N\left(A^{\mathrm{T}}\right) \subset R^{m}$.


## Stuff to know／understand：

Fundamental Theorem of Linear Algebra：
－Applies to any $m \times n$ matrix $A$ ．
－Symmetry of $A$ and $A^{\mathrm{T}}$ ．
－Column space $C(A) \subset R^{m}$ ．
－Left Nullspace $N\left(A^{T}\right) \subset R^{m}$ ．
－Row space $C\left(A^{\mathrm{T}}\right) \subset R^{n}$ ．
－（Right）Nullspace $N(A) \subset R^{n}$ ．

## Stuff to know／understand：

Fundamental Theorem of Linear Algebra：
－Applies to any $m \times n$ matrix $A$ ．
－Symmetry of $A$ and $A^{\mathrm{T}}$ ．
－Column space $C(A) \subset R^{m}$ ．
－Left Nullspace $N\left(A^{T}\right) \subset R^{m}$ ．
－ $\operatorname{dim} C(A)+\operatorname{dim} N\left(A^{\mathrm{T}}\right)=r+(m-r)=m$
－Row space $C\left(A^{\mathrm{T}}\right) \subset R^{n}$ ．
－（Right）Nullspace $N(A) \subset R^{n}$ ．

## Stuff to know／understand：

Fundamental Theorem of Linear Algebra：
－Applies to any $m \times n$ matrix $A$ ．
－Symmetry of $A$ and $A^{\mathrm{T}}$ ．
－Column space $C(A) \subset R^{m}$ ．
－Left Nullspace $N\left(A^{\mathrm{T}}\right) \subset R^{m}$ ．
－ $\operatorname{dim} C(A)+\operatorname{dim} N\left(A^{\mathrm{T}}\right)=r+(m-r)=m$
－Row space $C\left(A^{\mathrm{T}}\right) \subset R^{n}$ ．
－（Right）Nullspace $N(A) \subset R^{n}$ ．
－ $\operatorname{dim} C\left(A^{\mathrm{T}}\right)+\operatorname{dim} N(A)=r+(n-r)=n$

## Stuff to know／understand：

Fundamental Theorem of Linear Algebra：
－Applies to any $m \times n$ matrix $A$ ．
－Symmetry of $A$ and $A^{\mathrm{T}}$ ．
－Column space $C(A) \subset R^{m}$ ．
－Left Nullspace $N\left(A^{\mathrm{T}}\right) \subset R^{m}$ ．
－ $\operatorname{dim} C(A)+\operatorname{dim} N\left(A^{\mathrm{T}}\right)=r+(m-r)=m$
－Orthogonality：$C(A) \otimes N\left(A^{T}\right)=R^{m}$
－Row space $C\left(A^{\mathrm{T}}\right) \subset R^{n}$ ．
－（Right）Nullspace $N(A) \subset R^{n}$ ．
－ $\operatorname{dim} C\left(A^{\mathrm{T}}\right)+\operatorname{dim} N(A)=r+(n-r)=n$

## Stuff to know/understand:

Fundamental Theorem of Linear Algebra:

- Applies to any $m \times n$ matrix $A$.
- Symmetry of $A$ and $A^{\mathrm{T}}$.
- Column space $C(A) \subset R^{m}$.
- Left Nullspace $N\left(A^{T}\right) \subset R^{m}$.
- $\operatorname{dim} C(A)+\operatorname{dim} N\left(A^{\mathrm{T}}\right)=r+(m-r)=m$
- Orthogonality: $C(A) \otimes N\left(A^{T}\right)=R^{m}$
- Row space $C\left(A^{\mathrm{T}}\right) \subset R^{n}$.
- (Right) Nullspace $N(A) \subset R^{n}$.
- $\operatorname{dim} C\left(A^{\mathrm{T}}\right)+\operatorname{dim} N(A)=r+(n-r)=n$
- Orthogonality: $C\left(A^{T}\right) \otimes N(A)=R^{n}$


## Stuff to know/understand:

Finding four fundamental subspaces:

Words
Pictures


## Stuff to know／understand：

Finding four fundamental subspaces：
－Enough to find bases for subspaces．
Be able to reduce $A$ to $R$ ．
Identify pivot columns and free columns．
$\square$
Know that relationship between R＇s columns hold for A＇s columns．
 form a basis for $C(A)$ ．

## Stuff to know/understand:

Finding four fundamental subspaces:

- Enough to find bases for subspaces.
- Be able to reduce $A$ to $R$.

Identify pivat columns and free columns.
$\square$
Know that relationship between Fis columns hold for A's columns.


But find pivot columns in $B$, and same columns in $A$ form a hacis for $C(\Delta)$

## Stuff to know/understand:

Finding four fundamental subspaces:

- Enough to find bases for subspaces.
- Be able to reduce $A$ to $R$.
- Identify pivot columns and free columns.

Rankr of $A=\#$ pivot columns.


## Stuff to know/understand:

Finding four fundamental subspaces:

- Enough to find bases for subspaces.
- Be able to reduce $A$ to $R$.
- Identify pivot columns and free columns.
- Rank $r$ of $A=$ \# pivot columns.



## Stuff to know/understand:

Finding four fundamental subspaces:

- Enough to find bases for subspaces.
- Be able to reduce $A$ to $R$.
- Identify pivot columns and free columns.
- Rank $r$ of $A=$ \# pivot columns.
- Know that relationship between $R$ 's columns hold for A's columns.



## Stuff to know/understand:

Finding four fundamental subspaces:

- Enough to find bases for subspaces.
- Be able to reduce $A$ to $R$.
- Identify pivot columns and free columns.
- Rank $r$ of $A=$ \# pivot columns.
- Know that relationship between $R$ 's columns hold for A's columns.
- Warning: R's columns do not give a basis for $C(A)$



## Stuff to know／understand：

Finding four fundamental subspaces：
－Enough to find bases for subspaces．
－Be able to reduce $A$ to $R$ ．
－Identify pivot columns and free columns．
－Rank $r$ of $A=$ \＃pivot columns．
－Know that relationship between $R$＇s columns hold for A＇s columns．
－Warning：R＇s columns do not give a basis for $C(A)$
－But find pivot columns in $R$ ，and same columns in $A$ form a basis for $C(A)$ ．

## Stuff to know／understand：

More on bases for column and row space：

## Reduce $[A \mid b]$ where $b$ is general．

Find conditions on b＇s elements for a solution to $A \vec{x}=\vec{b}$ to exist

Basis for row space＝non－zero rows in $R$（easy！）
Alternate basis for column space＝non－zero rows in reduced form of $A^{1}$（easy！）


Ibe UNIVERSITY VERMONT $\left|\begin{array}{l}0 \\ 0 \\ 0\end{array}\right|$ つのく 7 of 16

## Stuff to know／understand：

More on bases for column and row space：
－Reduce $[A \mid \vec{b}]$ where $\vec{b}$ is general．
Find conditions on $\vec{b}$＇s elements for a solution to $A \vec{x}=\vec{b}$ to exist


Alternate basis for column space＝non－zero rows in reduced form of $A^{1}$（easy！）


Zibe of VERMONT

## Stuff to know/understand:

More on bases for column and row space:

- Reduce $[A \mid \vec{b}]$ where $\vec{b}$ is general.
- Find conditions on $\vec{b}$ 's elements for a solution to $A \vec{x}=\vec{b}$ to exist Alternate basis for column
reduced form of $A^{\mathrm{T}}$ (easy!)



## Stuff to know/understand:

More on bases for column and row space:

- Reduce $[A \mid \vec{b}]$ where $\vec{b}$ is general.
- Find conditions on $\vec{b}$ 's elements for a solution to $A \vec{x}=\vec{b}$ to exist $\rightarrow$ obtain basis for $C(A)$. Alternate basis for column s
reduced form of $A^{T}$ (easy!)



## Stuff to know/understand:

More on bases for column and row space:

- Reduce $[A \mid \vec{b}]$ where $\vec{b}$ is general.
- Find conditions on $\vec{b}$ 's elements for a solution to $A \vec{x}=\vec{b}$ to exist $\rightarrow$ obtain basis for $C(A)$.
- Basis for row space = non-zero rows in $R$ (easy!)
reduced form of $A^{T}$ (easy!)



## Stuff to know/understand:

More on bases for column and row space:

- Reduce $[A \mid \vec{b}]$ where $\vec{b}$ is general.
- Find conditions on $\vec{b}$ 's elements for a solution to $A \vec{x}=\vec{b}$ to exist $\rightarrow$ obtain basis for $C(A)$.
- Basis for row space = non-zero rows in $R$ (easy!)
- Alternate basis for column space = non-zero rows in reduced form of $A^{\mathrm{T}}$ (easy!)


## Stuff to know／understand：

## Bases for nullspaces，left and right：

## Basis for nullspace obtained by solving $A \vec{x}=0$

Always express pivot variables in terms of free variables．
－Free variables are unconstrained（can be any real number）

```
\# free variables \(=n-\) \# pivot variables \(=n-r=\operatorname{dim}\)
```

Similarly find basis for $N\left(A^{1}\right)$ by solving $A^{1} \vec{y}=0$ ．


Ibe NIVERSITY VERMONT $\left|\begin{array}{l}0 \\ 0 \\ 0\end{array}\right|$

## Stuff to know／understand：

Bases for nullspaces，left and right：
－Basis for nullspace obtained by solving $A \vec{x}=\overrightarrow{0}$
$\rightarrow$ Free variables are unconstrained（can be any real number）
\＃free variables $=n-$ \＃pivot variables $=n-r=\operatorname{dim}$

Similarly find basis for $N\left(A^{\mathrm{T}}\right)$ by solving $A^{\mathrm{T}} \vec{y}=0$ ．


The UNIVERSITY of VERMONT

っのく 8 of 16

## Stuff to know/understand:

Bases for nullspaces, left and right:

- Basis for nullspace obtained by solving $A \vec{x}=\overrightarrow{0}$
- Always express pivot variables in terms of free variables.
- Free variables are unconstrained (can be any real number)
$\square$
$\rightarrow$ Similarly find basis for $N\left(A^{\mathrm{T}}\right)$ by solving $A^{\mathrm{T}} \vec{y}=0$.


UNIVERSITY of VERMONT
$\left|\begin{array}{l}0 \\ 0 \\ 0\end{array}\right|$

๑aく 8 of 16

## Stuff to know/understand:

Bases for nullspaces, left and right:

- Basis for nullspace obtained by solving $A \vec{x}=\overrightarrow{0}$
- Always express pivot variables in terms of free variables.
- Free variables are unconstrained (can be any real number)
$\rightarrow$ \# free variables $=n$ - \# pivot variables $=n-r=\operatorname{dim}$


## Stuff to know／understand：

Bases for nullspaces，left and right：
－Basis for nullspace obtained by solving $A \vec{x}=\overrightarrow{0}$
－Always express pivot variables in terms of free variables．
－Free variables are unconstrained（can be any real number）
－\＃free variables $=n$－\＃pivot variables $=n-r=\operatorname{dim}$ $N(A)$ ．

## Stuff to know/understand:

Bases for nullspaces, left and right:

- Basis for nullspace obtained by solving $A \vec{x}=\overrightarrow{0}$
- Always express pivot variables in terms of free variables.
- Free variables are unconstrained (can be any real number)
- \# free variables $=n$ - \# pivot variables $=n-r=\operatorname{dim}$ $N(A)$.
- Similarly find basis for $N\left(A^{\mathrm{T}}\right)$ by solving $A^{\mathrm{T}} \vec{y}=\overrightarrow{0}$.


## Stuff to know/understand:

Number of solutions to $A \vec{x}=\vec{b}$ :

> If $b \notin C(A)$, there are no solutions.
> 2. If $\vec{b} \in C(A)$ there is either one unique solution or infinitely many solutions.

## Stuff to know/understand:

Number of solutions to $A \vec{x}=\vec{b}$ :

1. If $\vec{b} \notin C(A)$, there are no solutions.
2. If $b \in C(A)$ there is either one unique solution or infinitely many solutions.

## Stuff to know/understand:

Number of solutions to $A \vec{x}=\vec{b}$ :

1. If $\vec{b} \notin C(A)$, there are no solutions.
2. If $\vec{b} \in C(A)$ there is either one unique solution or infinitely many solutions.


## Stuff to know/understand:

Number of solutions to $A \vec{x}=\vec{b}$ :

1. If $\vec{b} \notin C(A)$, there are no solutions.
2. If $\vec{b} \in C(A)$ there is either one unique solution or infinitely many solutions.

- Number of solutions now depends entirely on $N(A)$.
$f \operatorname{dim} N(A)=n-r=0$, then there is one solution.


## Stuff to know/understand:

Number of solutions to $A \vec{x}=\vec{b}$ :

1. If $\vec{b} \notin C(A)$, there are no solutions.
2. If $\vec{b} \in C(A)$ there is either one unique solution or infinitely many solutions.

- Number of solutions now depends entirely on $N(A)$.
- If $\operatorname{dim} N(A)=n-r>0$, then there are infinitely many solutions.
then there is one solution.


## Stuff to know/understand:

Number of solutions to $A \vec{x}=\vec{b}$ :

1. If $\vec{b} \notin C(A)$, there are no solutions.
2. If $\vec{b} \in C(A)$ there is either one unique solution or infinitely many solutions.

- Number of solutions now depends entirely on $N(A)$.
- If $\operatorname{dim} N(A)=n-r>0$, then there are infinitely many solutions.
- If $\operatorname{dim} N(A)=n-r=0$, then there is one solution.


## Projections:

- $\vec{e}=$ that part of $\vec{b}$ that is orthogonal to the line. Understand generalization to projection onto subspaces.

Understand construction and use of subspace projection operator $P$ :

$$
\vec{P}=A\left(A^{\mathrm{T}} A\right)^{-1} A^{\mathrm{T}},
$$

where A's columns form a subspace basis.


The
UNIVERSITY
aa^ 10 of 16

## Projections：

－Understand how to project a vector $\vec{b}$ onto a line in direction of $\vec{a}$ ．
$\vec{e}=$ that part of $\vec{b}$ that is orthogonal to the line．
Understand gemeralization to proiection onto subspaces．

Understand construction and use of subspace projection operator $P$ ：
where A＇s columns form a subspace basis．

ITN UNIVERSITY of VERMONT

## Projections：

－Understand how to project a vector $\vec{b}$ onto a line in direction of $\vec{a}$ ．
－$\vec{b}=\vec{p}+\vec{e}$
$\vec{p}=$ that part of $b$ that les in the line：

$$
\vec{p}=\frac{\overrightarrow{a^{T}} \vec{b}}{\vec{a}^{T} \vec{a}} \vec{a}\left(=\frac{\vec{a}^{T}}{\vec{a}^{T}} \vec{a} \vec{b}\right)
$$

$\vec{e}=$ that part of $\vec{b}$ that is orthogonal to the line． Understand generalization to projection onto subspaces．

Understand construction and use of subspace projection operator $P$ ：
where A＇s columns form a subspace basis．

The UNIVERSITY of VERMONT

## Projections:

- Understand how to project a vector $\vec{b}$ onto a line in direction of $\vec{a}$.
- $\vec{b}=\vec{p}+\vec{e}$
- $\vec{p}=$ that part of $\vec{b}$ that lies in the line:

$$
\vec{p}=\frac{\vec{a}^{T} \vec{b}}{\vec{a}^{T} \vec{a}} \vec{a}\left(=\frac{\vec{a}}{\vec{a} \vec{a}^{T}} \overrightarrow{\vec{a}^{T}} \vec{a}\right)
$$

$\vec{e}=$ that part of $\vec{b}$ that is orthogonal to the line.


$$
\vec{p}=\frac{\vec{a}^{T} \vec{b}}{\vec{a}^{T} \vec{a}} \vec{a}\left(=\frac{\vec{a} \vec{a}^{T}}{\vec{a}^{T} \vec{b}} \vec{a}\right)
$$

$\vec{e}=$ that part of $\vec{b}$ that is orthogonal to the line.
Understand generalization to pro ection onto
subspaces.
Understand construction and use of subspace
projection operator $P$ :


UNIVERSITY of VERMONT

## Projections:

- Understand how to project a vector $\vec{b}$ onto a line in direction of $\vec{a}$.
- $\vec{b}=\vec{p}+\vec{e}$
- $\vec{p}=$ that part of $\vec{b}$ that lies in the line:

$$
\vec{p}=\frac{\vec{a}^{\mathrm{T}} \vec{b}}{\vec{a}^{\mathrm{T}} \vec{a}} \vec{a}\left(=\frac{\vec{a} \vec{a}^{\mathrm{T}}}{\vec{a}^{\mathrm{a}} \vec{a} \vec{b}}\right)
$$

- $\vec{e}=$ that part of $\vec{b}$ that is orthogonal to the line.



## Projections:

- Understand how to project a vector $\vec{b}$ onto a line in direction of $\vec{a}$.
- $\vec{b}=\vec{p}+\vec{e}$
- $\vec{p}=$ that part of $\vec{b}$ that lies in the line:

$$
\vec{p}=\frac{\vec{a}^{T} \vec{b}}{\vec{a}^{T} \vec{a}} \vec{a}\left(=\frac{\vec{a} \vec{a}^{T}}{\vec{a}^{T} \vec{a}^{T} \vec{b}}\right)
$$

- $\vec{e}=$ that part of $\vec{b}$ that is orthogonal to the line.
- Understand generalization to projection onto subspaces.



## Projections:

- Understand how to project a vector $\vec{b}$ onto a line in direction of $\vec{a}$.
- $\vec{b}=\vec{p}+\vec{e}$
- $\vec{p}=$ that part of $\vec{b}$ that lies in the line:

$$
\vec{p}=\frac{\vec{a}^{T} \vec{b}}{\vec{a}^{T} \vec{a}} \vec{a}\left(=\frac{\vec{a}}{\vec{a} \vec{a}^{T}} \overrightarrow{\vec{a}^{T}} \vec{a}\right)
$$

- $\vec{e}=$ that part of $\vec{b}$ that is orthogonal to the line.
- Understand generalization to projection onto subspaces.
- Understand construction and use of subspace projection operator $P$ :

$$
\vec{P}=A\left(A^{\mathrm{T}} A\right)^{-1} A^{\mathrm{T}},
$$

where A's columns form a subspace basis.

Review for Exam 2
Words
Pictures


UNDIVERSITY of VERMONT

## Stuff to know／understand

Normal equation for $A \vec{x}=\vec{b}$ ：

## If $\vec{b} \notin C(A)$ ，project $\vec{b}$ onto $C(A)$

Write projection of $\vec{b}$ as $\vec{p}$ ．
Know $\vec{p} \in C(A)$ so $\exists \vec{x}_{*}$ such that $A \vec{x}_{*}=\vec{p}$ ．
Error vector must be orthogonal to column space so $A^{\mathrm{T}} \vec{e}=A^{\mathrm{T}}(\vec{b}-\vec{p})=\overrightarrow{0}$ ．
－Rearrange：

－Since $A \vec{x}_{*}=\vec{p}$ ，we end up with
－This is linear algebra＇s normal equation； $\vec{x}$ ，is our best solution to $A \vec{x}=\vec{b}$ ．


The
UNIVERSITY VERMONT

## Stuff to know／understand

Normal equation for $A \vec{x}=\vec{b}$ ：
－If $\vec{b} \notin C(A)$ ，project $\vec{b}$ onto $C(A)$ ．
$\square$
Error vector must be orthogonal to column space so $A^{\mathrm{T}} \vec{e}=A^{\mathrm{T}}(\vec{b}-\vec{p})=\overrightarrow{0}$ ．
－Rearrange：
－Since $A \vec{x}_{*}=\vec{p}$ ，we end up with
－This is linear algebra＇s normal equation； $\vec{x}$ ，is our best solution to $A \vec{x}=\vec{b}$ ．

The UNIVERSITY V VERMONT

つの『 11 of 16

## Stuff to know／understand

Normal equation for $A \vec{x}=\vec{b}$ ：
－If $\vec{b} \notin C(A)$ ，project $\vec{b}$ onto $C(A)$ ．
－Write projection of $\vec{b}$ as $\vec{p}$ ．


Error vector must be orthogonal to column space so $A^{\mathrm{T}} \dot{e}=A^{\mathrm{T}}(\vec{b}-\vec{p})=0$ ．
－Rearrange：

$\square$

$\qquad$

## Stuff to know/understand

Normal equation for $A \vec{x}=\vec{b}$ :

- If $\vec{b} \notin C(A)$, project $\vec{b}$ onto $C(A)$.
- Write projection of $\vec{b}$ as $\vec{p}$.
- Know $\vec{p} \in C(A)$ so $\exists \vec{x}_{*}$ such that $A \vec{x}_{*}=\vec{p}$.


## Stuff to know/understand

Normal equation for $A \vec{x}=\vec{b}$ :

- If $\vec{b} \notin C(A)$, project $\vec{b}$ onto $C(A)$.
- Write projection of $\vec{b}$ as $\vec{p}$.
- Know $\vec{p} \in C(A)$ so $\exists \vec{x}_{*}$ such that $A \vec{x}_{*}=\vec{p}$.
- Error vector must be orthogonal to column space so $A^{\mathrm{T}} \vec{e}=A^{\mathrm{T}}(\vec{b}-\vec{p})=\overrightarrow{0}$.


## Stuff to know/understand

Normal equation for $A \vec{x}=\vec{b}$ :

- If $\vec{b} \notin C(A)$, project $\vec{b}$ onto $C(A)$.
- Write projection of $\vec{b}$ as $\vec{p}$.
- Know $\vec{p} \in C(A)$ so $\exists \vec{x}_{*}$ such that $A \vec{x}_{*}=\vec{p}$.
- Error vector must be orthogonal to column space so $A^{\mathrm{T}} \vec{e}=A^{\mathrm{T}}(\vec{b}-\vec{p})=\overrightarrow{0}$.
- Rearrange:

$$
A^{\mathrm{T}} \vec{p}=A^{\mathrm{T}} \vec{b}
$$

## Stuff to know/understand

Normal equation for $A \vec{x}=\vec{b}$ :

- If $\vec{b} \notin C(A)$, project $\vec{b}$ onto $C(A)$.
- Write projection of $\vec{b}$ as $\vec{p}$.
- Know $\vec{p} \in C(A)$ so $\exists \vec{x}_{*}$ such that $A \vec{x}_{*}=\vec{p}$.
- Error vector must be orthogonal to column space so $A^{\mathrm{T}} \vec{e}=A^{\mathrm{T}}(\vec{b}-\vec{p})=\overrightarrow{0}$.
- Rearrange:

$$
A^{\mathrm{T}} \vec{p}=A^{\mathrm{T}} \vec{b}
$$

- Since $A \vec{x}_{*}=\vec{p}$, we end up with

$$
A^{\mathrm{T}} A \vec{x}_{*}=A^{\mathrm{T}} \vec{b} .
$$

## Stuff to know/understand

Normal equation for $A \vec{x}=\vec{b}$ :

- If $\vec{b} \notin C(A)$, project $\vec{b}$ onto $C(A)$.
- Write projection of $\vec{b}$ as $\vec{p}$.
- Know $\vec{p} \in C(A)$ so $\exists \vec{x}_{*}$ such that $A \vec{x}_{*}=\vec{p}$.
- Error vector must be orthogonal to column space so $A^{\mathrm{T}} \vec{e}=A^{\mathrm{T}}(\vec{b}-\vec{p})=\overrightarrow{0}$.
- Rearrange:

$$
A^{\mathrm{T}} \vec{p}=A^{\mathrm{T}} \vec{b}
$$

- Since $A \vec{x}_{*}=\vec{p}$, we end up with

$$
A^{\mathrm{T}} A \vec{\chi}_{*}=A^{\mathrm{T}} \vec{b} .
$$

- This is linear algebra's normal equation; $\vec{x}_{*}$ is our best solution to $A \vec{x}=\vec{b}$.

${ }^{\text {Tho }}$ UNIVERSITY of VERMONT


## The symmetry of $A \vec{x}=\vec{b}$ and $A^{\mathrm{T}} \vec{y}=\vec{c}$ :



The
UNIVERSITY of VERMONT

How $A \vec{x}=\vec{b}$ works:

Best solution $\vec{x}_{*}$ when $\vec{b}=\vec{p}+\vec{e}$ :

The fourfold ways of $A \vec{x}=\vec{b}$ ：

| case | example $R$ | big picture | \＃ solutions |
| :---: | :---: | :---: | :---: |
| $\begin{aligned} m & =r \\ n & =r \end{aligned}$ | $\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$ |  | 1 always |
| $\begin{gathered} m=r \\ n>r \end{gathered}$ | $\left[\begin{array}{lll}1 & 0 & 1 \\ 0 & 1 & { }_{2}\end{array}\right]$ |  | $\infty$ always |
| $\begin{gathered} m>r \\ n=r \end{gathered}$ | $\left[\begin{array}{ll}1 & 0 \\ 0 & 1 \\ 0 & 0\end{array}\right]$ |  | 0 or 1 |
| $\begin{aligned} m & >r \\ n & >r \end{aligned}$ | $\left[\begin{array}{lll}1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$ |  | 0 or $\infty$ |



The UNIVERSITY of VERMONT

