Optimal Supply Networks

Complex Networks, CSYS/MATH 303, Spring, 2010

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Outline

Optimal supply networks

What's the best way to distribute stuff?

- Stuff = medical services, energy, people,
- Some fundamental network problems:
  1. Distribute stuff from a single source to many sinks
  2. Distribute stuff from many sources to many sinks
  3. Redistribute stuff between nodes that are both sources and sinks
- Supply and Collection are equivalent problems

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Single source optimal supply

Basic Q for distribution/supply networks:

- How does flow behave given cost:

$$C = \sum_j I_j^{\gamma} Z_j$$

  where
  I_j = current on link j
  and
  Z_j = link j's impedance?
- Example: $\gamma = 2$ for electrical networks.

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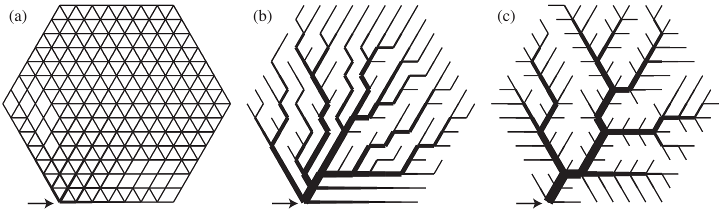


(a) $\gamma > 1$: Braided (bulk) flow

(b) $\gamma < 1$: Local minimum: Branching flow

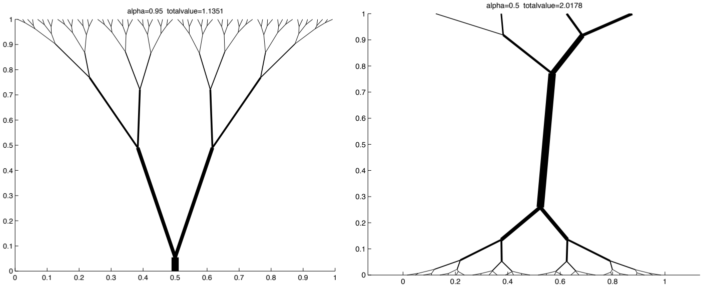
(c) $\gamma < 1$: Global minimum: Branching flow

From Bohn and Magnasco [3]

See also Banavar et al. [1]

Single source optimal supply

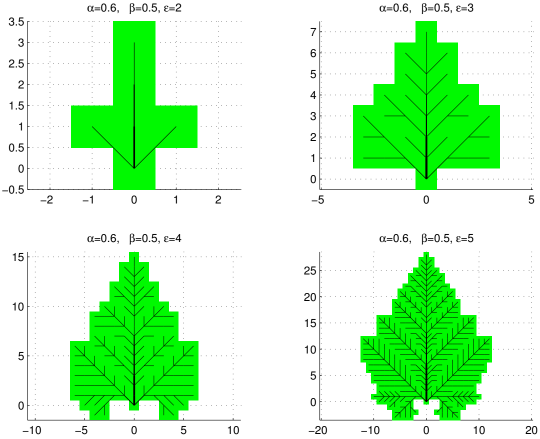
Optimal paths related to transport (Monge) problems:



Xia (2003) [27]

Growing networks:

FIGURE 1. $\alpha = 0.6, \beta = 0.5$



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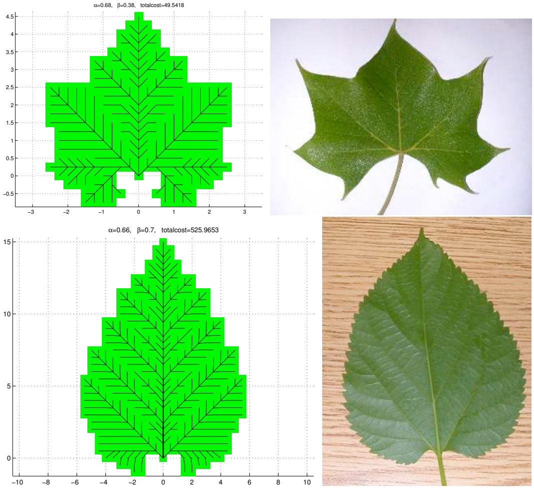
Growing networks:

FIGURE 3. A maple leaf



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Single source optimal supply

An immensely controversial issue...

- The form of river networks and blood networks: optimal or not? [25, 2, 5, 4]

Two observations:

- Self-similar networks appear everywhere in nature for single source supply/single sink collection.
- Real networks differ in details of scaling but reasonably agree in scaling relations.

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River network models

Optimality:

- Optimal channel networks [15]
- Thermodynamic analogy [16]

versus...

Randomness:

- Scheidegger's directed random networks
- Undirected random networks

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Optimization approaches

Cardiovascular networks:

- Murray's law (1926) connects branch radii at forks: [13, 12, 14]

$$r_0^3 = r_1^3 + r_2^3$$

  where r_0 = radius of main branch
  and r_1 and r_2 are radii of sub-branches.
- See D'Arcy Thompson's "On Growth and Form" for background inspiration [20, 21].
- Calculation assumes Poiseuille flow (⊞).
- Holds up well for outer branchings of blood networks.
- Also found to hold for trees [14, 10, 11].
- Use hydraulic equivalent of Ohm's law:

$$\Delta p = \Phi Z \Leftrightarrow V = IR$$

  where Δp = pressure difference, Φ = flux.

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Cardiovascular networks:

- Fluid mechanics: Poiseuille impedance (⊞) for smooth flow in a tube of radius r and length ℓ:

$$Z = \frac{8\eta\ell}{\pi r^4}$$

  where η = dynamic viscosity (⊞) (units: $ML^{-1}T^{-1}$).
- Power required to overcome impedance:

$$P_{\mathrm{drag}} = \Phi\Delta p = \Phi^2 Z.$$

- Also have rate of energy expenditure in maintaining blood:

$$P_{\mathrm{metabolic}} = cr^2\ell$$

  where c is a metabolic constant.

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Optimization approaches

Aside on P_{drag}

- Work done = $F \cdot d$ = energy transferred by force F
- Power = P = rate work is done = $F \cdot v$
- Δp = Force per unit area
- Φ = Volume per unit time
  = cross-sectional area · velocity
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Murray's law:

- Total power (cost):

$$P = P_{\text{drag}} + P_{\text{metabolic}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell$$

- Observe power increases linearly with ℓ
- But r's effect is nonlinear:
  - increasing r makes flow easier but increases metabolic cost (as r^2)
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$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + c r^2 \ell\right)$$

$$= -4\Phi^2 \frac{8\eta\ell}{\pi r^5} + c2r\ell = 0$$

- Rearrange/cancel/slap:

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- So we now have:

$$\Phi = kr^3$$

- Flow rates at each branching have to add up (else our organism is in serious trouble...):

$$\Phi_0 = \Phi_1 + \Phi_2$$

where again 0 refers to the main branch and 1 and 2 refers to the offspring branches

- All of this means we have a groovy cube-law:

$$\boxed{r_0^3 = r_1^3 + r_2^3}$$

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Murray meets Tokunaga:

- Φ_ω = volume rate of flow into an order ω vessel segment
- Tokunaga picture:

$$\Phi_\omega = 2\Phi_{\omega-1} + \sum_{k=1}^{\omega-1} T_k \Phi_{\omega-k}$$

- Using $\phi_\omega = k r_\omega^3$

$$r_\omega^3 = 2r_{\omega-1}^3 + \sum_{k=1}^{\omega-1} T_k r_{\omega-k}^3$$

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- Find R_r^3 satisfies same equation as R_n and R_v (v is for volume):

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- Is there more we could do here to constrain the Horton ratios and Tokunaga constants?

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- Isometry: $V_\omega \propto \ell_\omega^3$
- Gives

$$R_\ell^3 = R_v = R_n$$

- We need one more constraint...
- West et al (1997) [25] achieve similar results following Horton's laws.
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Geometric argument

- Consider one source supplying many sinks in a volume V d-dim. region in a D-dim. ambient space.
- Assume sinks are invariant.
- Assume $\rho = \rho(V)$, i.e., ρ may vary with region's volume V.
- See network as a bundle of virtual vessels:
- Q: how does the number of sustainable sinks N_{sinks} scale with volume V for the most efficient network design?
- Or: what is the highest α for $N_{\text{sinks}} \propto V^{\alpha}$?

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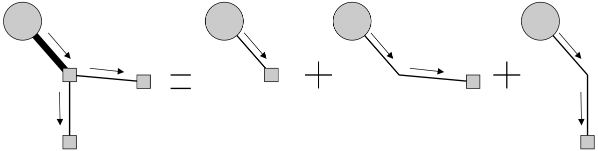

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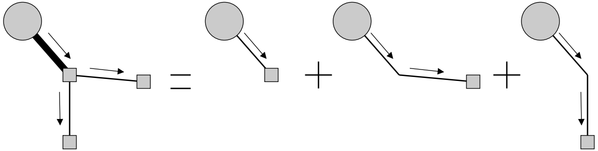

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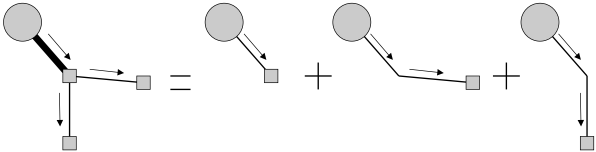

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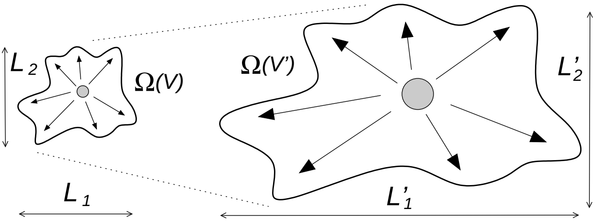
Geometric argument

- Allometrically growing regions:



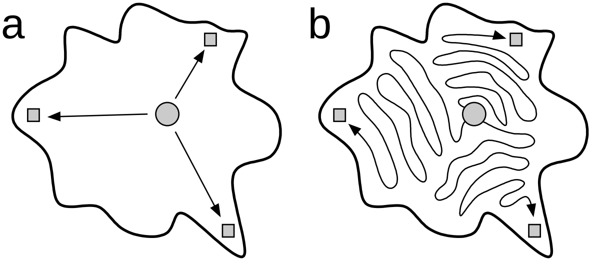
- Have d length scales which scale as

$$L_i \propto V^{\gamma_i} \text{ where } \gamma_1 + \gamma_2 + \ldots + \gamma_d = 1.$$

- For isometric growth, $\gamma_i = 1/d$.
- For allometric growth, we must have at least two of the $\{\gamma_i\}$ being different

Geometric argument

- Best and worst configurations (Banavar et al.)



- Rather obviously:
  min $V_{\text{net}} \propto \sum$ distances from source to sinks.

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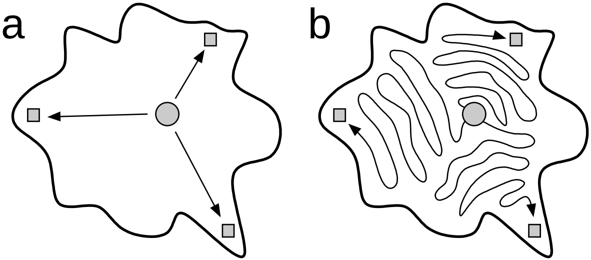

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Minimal network volume:

Real supply networks are close to optimal:

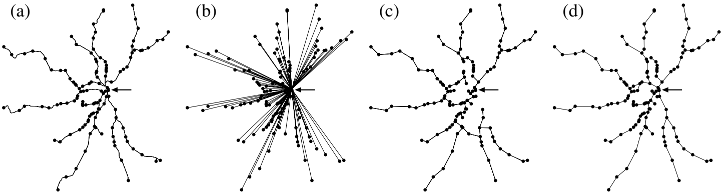

Figure 1. (a) Commuter rail network in the Boston area. The arrow marks the assumed root of the network. (b) Star graph. (c) Minimum spanning tree. (d) The model of equation (3) applied to the same set of stations.

(2006) Gastner and Newman [8]: "Shape and efficiency in spatial distribution networks"

Minimal network volume:

Add one more element:

- Vessel cross-sectional area may vary with distance from the source.
- Flow rate increases as cross-sectional area decreases.
- e.g., a collection network may have vessels tapering as they approach the central sink.
- Find that vessel volume v must scale with vessel length ℓ to affect overall system scalings.
- Consider vessel radius $r \propto (\ell + 1)^{-\epsilon}$, tapering from $r = r_{\text{max}}$ where $\epsilon \geq 0$.
- Gives $v \propto \ell^{1-2\epsilon}$ if $\epsilon < 1/2$
- Gives $v \propto 1 - \ell^{-(2\epsilon-1)} \rightarrow 1$ for large ℓ if $\epsilon > 1/2$
- Previously, we looked at $\epsilon = 0$ only.

Minimal network volume:

For $0 \leq \epsilon < 1/2$, approximate network volume by integral over region:

$$\min V_{\text{net}} \propto \int_{\Omega_{d,D}(V)} \rho \, ||\vec{x}||^{1-2\epsilon} \, \mathrm{d}\vec{x}$$

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For $0 \leq \epsilon < 1/2$:

- $\boxed{\min V_{\mathrm{net}} \propto \rho V^{1+\gamma_{\max}(1-2\epsilon)}}$
- If scaling is isometric, we have $\gamma_{\max} = 1/d$:

$$\min V_{\mathrm{net/iso}} \propto \rho V^{1+(1-2\epsilon)/d}$$

- If scaling is allometric, we have $\gamma_{\max} = \gamma_{\mathrm{allo}} > 1/d$: and

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- Isometrically growing volumes require less network volume than allometrically growing volumes:

$$\frac{\min V_{\mathrm{net/iso}}}{\min V_{\mathrm{net/allo}}} \to 0 \text{ as } V \to \infty$$

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Limits to scaling

- Can argue that ϵ must effectively be 0 for real networks over large enough scales.
- Limit to how fast material can move, and how small material packages can be.
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Optimal branching
Murray's law
Murray meets Tokunaga
Single Source
Geometric argument
Blood networks
River networks
Distributed Sources
Facility location
Size-density law
Cartograms
A reasonable derivation
Global redistribution networks
Public versus Private
References

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- The exponent $\alpha = 2/3$ works for all birds and mammals up to 10–30 kg
- For mammals $>$ 10–30 kg, maybe we have a new scaling regime
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River networks

- View river networks as collection networks.
- Many sources and one sink.
- ϵ?
- Assume ρ is constant over time and $\epsilon = 0$:

$$V_{\text{net}} \propto \rho V^{(d+1)/d} = \text{constant} \times V^{3/2}$$

- Network volume grows faster than basin 'volume' (really area).
- It's all okay:
  Landscapes are d=2 surfaces living in D=3 dimension.
- Streams can grow not just in width but in depth...
- If $\epsilon > 0$, V_{net} will grow more slowly but 3/2 appears to be confirmed from real data.

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Many sources, many sinks

How do we distribute sources?

- Focus on 2-d (results generalize to higher dimensions)
- Sources = hospitals, post offices, pubs, ...
- Key problem: How do we cope with uneven population densities?
- Obvious: if density is uniform then sources are best distributed uniformly
- Which lattice is optimal? The hexagonal lattice
  Q1: How big should the hexagons be?
- Q2: Given population density is uneven, what do we do?
- We'll follow work by Stephan [18, 19], Gastner and Newman (2006) [7], Um *et al.* [23] and work cited by them.

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Optimal source allocation

Solidifying the basic problem

- Given a region with some population distribution ρ, most likely uneven.
- Given resources to build and maintain N facilities.
- Q: How do we locate these N facilities so as to minimize the average distance between an individual's residence and the nearest facility?

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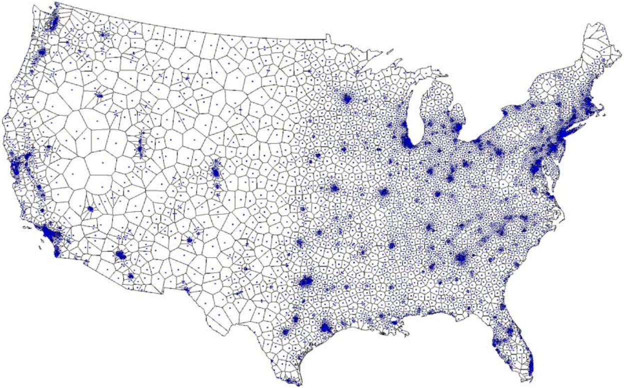
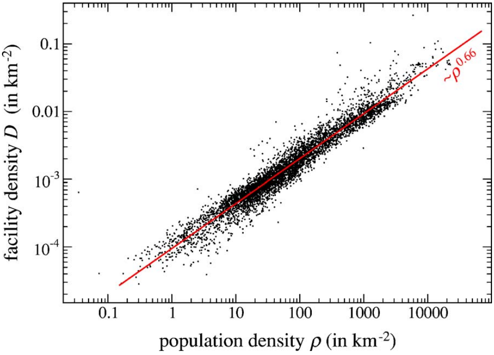


From Gastner and Newman (2006) [7]

- Approximately optimal location of 5000 facilities.
- Based on 2000 Census data.
- Simulated annealing + Voronoi tessellation.

Optimal source allocation



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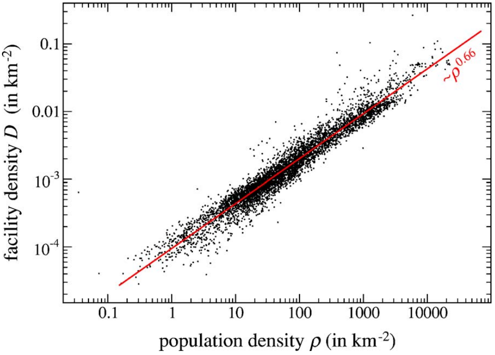

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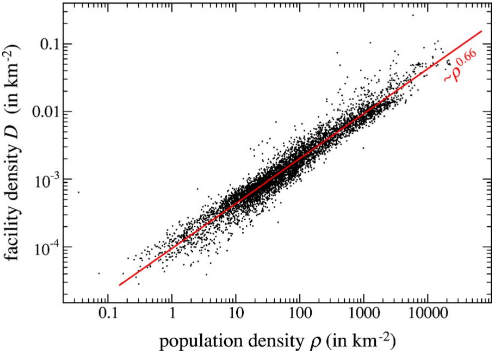

From Gastner and Newman (2006) [7]

- Optimal facility density D vs. population density ρ.
- Fit is $D \propto \rho^{0.66}$ with $r^2 = 0.94$.
- Looking good for a 2/3 power...

Outline

Optimal source allocation

Size-density law:

- $$D \propto \rho^{2/3}$$
- Why?
- Again: Different story to branching networks where there was either one source or one sink.
- Now sources & sinks are distributed throughout region...

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- We first examine Stephan's treatment (1977) [18, 19]
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- Consider a region of area A and population P with a single functional center that everyone needs to access every day.
- Build up a general cost function based on time expended to access and maintain center.
- Write average travel distance to center as $\bar{d}$ and assume average speed of travel is $\bar{v}$.
- Assume isometry: average travel distance $\bar{d}$ will be on the length scale of the region which is $\sim A^{1/2}$
- Average time expended per person in accessing facility is therefore

$$\bar{d}/\bar{v} = cA^{1/2}/\bar{v}$$

where c is an unimportant shape factor.

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- Next assume facility requires regular maintenance (person-hours per day)
- Call this quantity τ
- If burden of mainenance is shared then average cost per person is τ/P where P = population.
- Replace P by ρA where ρ is density.
- Total average time cost per person:

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- Differentiating...

$$\frac{\partial T}{\partial A} = \frac{\partial}{\partial A}\left(cA^{1/2}/\bar{v} + \tau/(\rho A)\right)$$

$$= \frac{c}{2\bar{v}A^{1/2}} - \frac{\tau}{\rho A^2} = 0$$

- Rearrange:

$$A = \left(\frac{2\bar{v}\tau}{c\rho}\right)^{2/3} \propto \rho^{-2/3}$$

- # facilities per unit area $\propto$

$$A^{-1} \propto \rho^{2/3}$$

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Optimal source allocation

An issue:

- Maintenance (τ) is assumed to be independent of population and area (P and A)

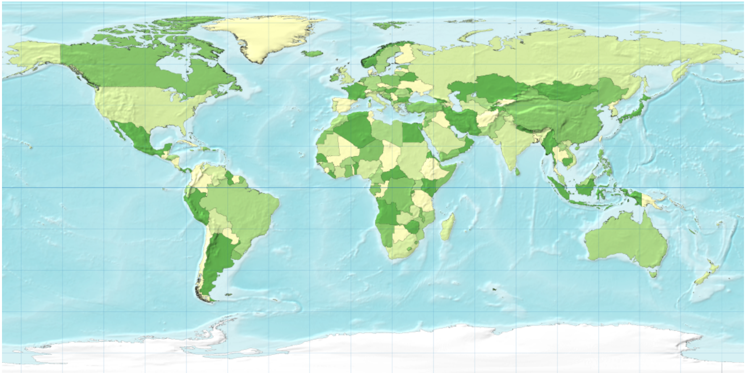
Optimal source allocation

Stephan's online book
"The Division of Territory in Society" is here (⊞).

Outline

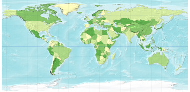
Cartograms

Standard world map:



Cartograms

Cartogram of countries 'rescaled' by population:



Cartograms

Diffusion-based cartograms:

- Idea of cartograms is to distort areas to more accurately represent some local density ρ (e.g. population).
- Many methods put forward—typically involve some kind of physical analogy to spreading or repulsion.
- Algorithm due to Gastner and Newman (2004) [6] is based on standard diffusion:

$$\nabla^2 \rho - \frac{\partial \rho}{\partial t} = 0.$$

- Allow density to diffuse and trace the movement of individual elements and boundaries.
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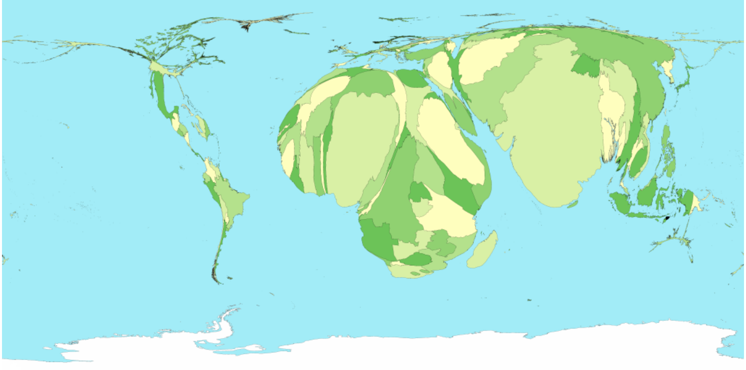
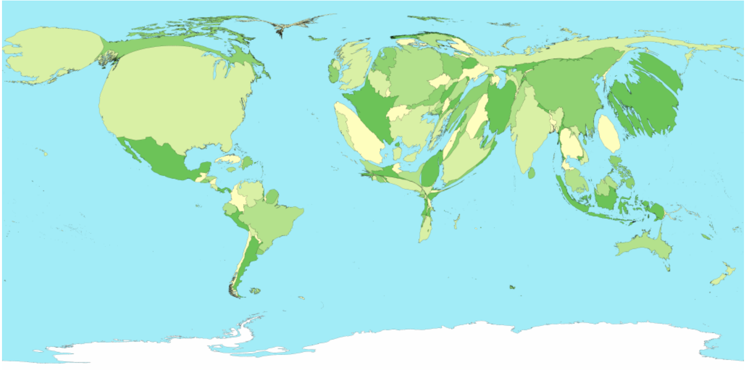


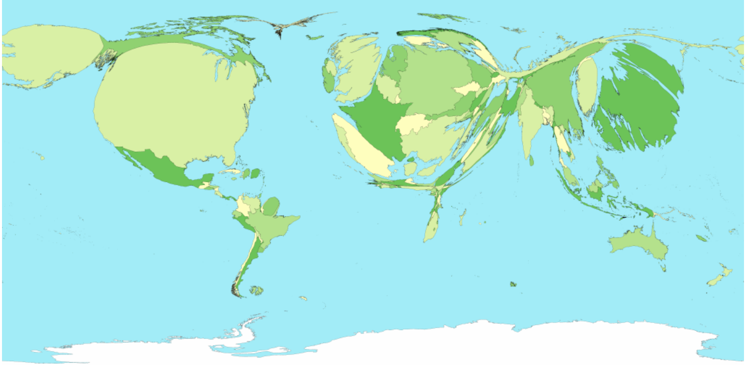
Cartograms

Energy consumption:



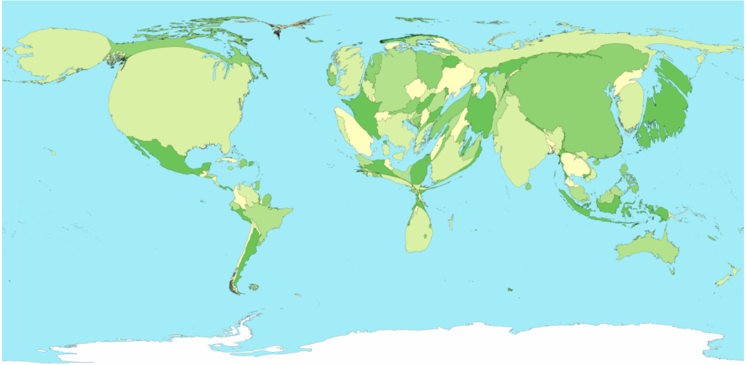
Cartograms

Gross domestic product:



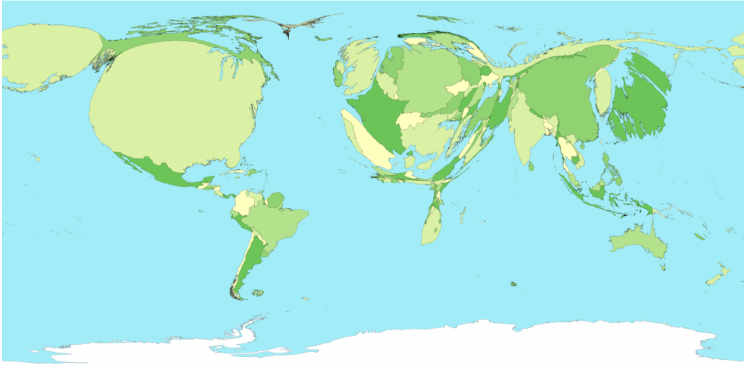
Cartograms

Greenhouse gas emissions:



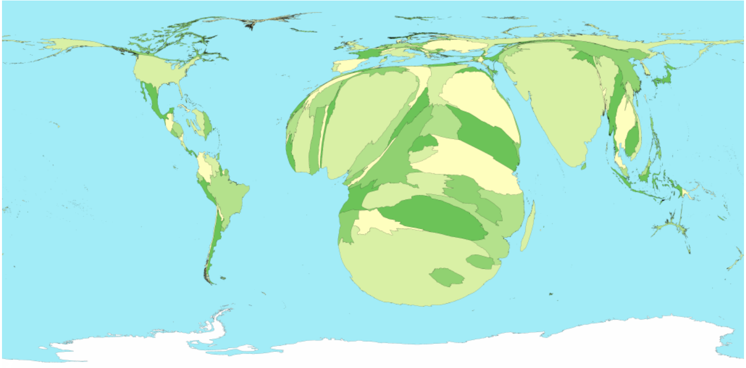
Cartograms

Spending on healthcare:



Cartograms

People living with HIV:



Cartograms

- The preceding sampling of Gastner & Newman's cartograms lives here (⊞).
- A larger collection can be found at worldmapper.org (⊞).

WORLDMAPPER The world as you've never seen it before

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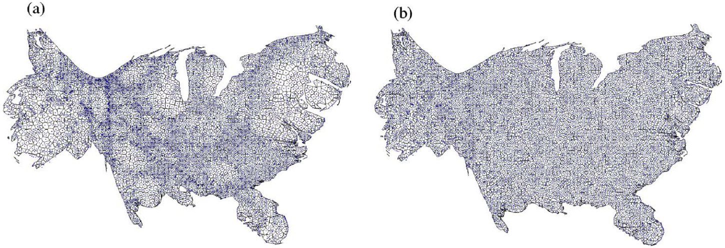

- Left: population density-equalized cartogram.
- Right: (population density)$^{2/3}$-equalized cartogram.
- Facility density is uniform for $\rho^{2/3}$ cartogram.

From Gastner and Newman (2006) [7]

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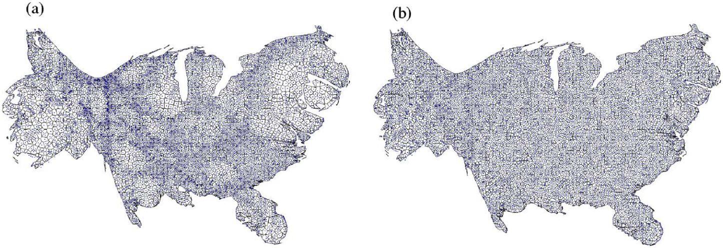

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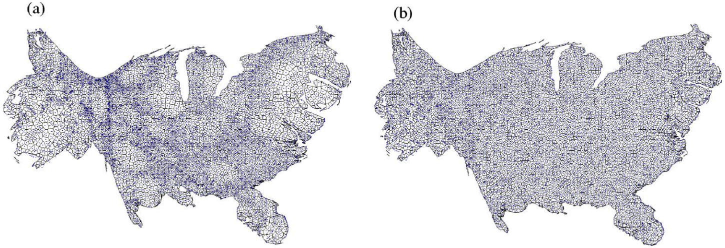

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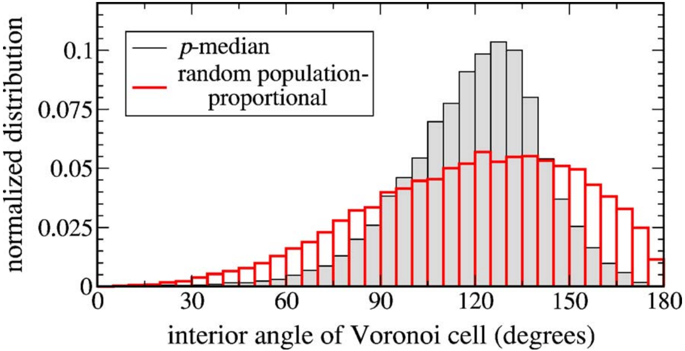

From Gastner and Newman (2006) [7]

- Cartogram's Voronoi cells are somewhat hexagonal.

Outline

Size-density law

Deriving the optimal source distribution:

- Basic idea: Minimize the average distance from a random individual to the nearest facility. [7]
- Assume given a fixed population density ρ defined on a spatial region Ω.
- Formally, we want to find the locations of n sources $\{\vec{x}_1, \ldots, \vec{x}_n\}$ that minimizes the cost function

$$F(\{\vec{x}_1, \ldots, \vec{x}_n\}) = \int_{\Omega} \rho(\vec{x}) \min_i ||\vec{x} - \vec{x}_i|| \mathrm{d}\vec{x}\,.$$

- Also known as the p-median problem.
- Not easy... in fact this one is an NP-hard problem. [7]
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Approximations:

- For a given set of source placements $\{\vec{x}_1, \ldots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells (⊞), one per source.
- Define $A(\vec{x})$ as the area of the Voronoi cell containing $\vec{x}$.
- As per Stephan's calculation, estimate typical distance from $\vec{x}$ to the nearest source (say i) as

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where c_i is a shape factor for the ith Voronoi cell.
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  where c_i is a shape factor for the ith Voronoi cell.
- Approximate c_i as a constant c.

Size-density law

Approximations:

- For a given set of source placements $\{\vec{x}_1, \ldots, \vec{x}_n\}$, the region Ω is divided up into Voronoi cells (⊞), one per source.
- Define $A(\vec{x})$ as the area of the Voronoi cell containing $\vec{x}$.
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Carrying on:

- The cost function is now

$$F = c \int_{\Omega} \rho(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} \,.$$

- We also have that the constraint that Voronoi cells divide up the overall area of Ω: $\sum_{i=1}^{n} A(\vec{x}_i) = A_{\Omega}$.
- Sneakily turn this into an integral constraint:

$$\int_{\Omega} \frac{\mathrm{d}\vec{x}}{A(\vec{x})} = n.$$

- Within each cell, $A(\vec{x})$ is constant.
- So... integral over each of the n cells equals 1.

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Now a Lagrange multiplier story:

- By varying $\{\vec{x}_1, \ldots, \vec{x}_n\}$, minimize

$$G(A) = c \int_\Omega \rho(\vec{x}) A(\vec{x})^{1/2} \mathrm{d}\vec{x} - \lambda \left(n - \int_\Omega [A(\vec{x})]^{-1} \mathrm{d}\vec{x} \right)$$

- Next compute $\delta G / \delta A$, the functional derivative (⊞) of the functional $G(A)$.
- This gives

$$\int_\Omega \left[\frac{c}{2} \rho(\vec{x}) A(\vec{x})^{-1/2} - \lambda [A(\vec{x})]^{-2} \right] \mathrm{d}\vec{x} = 0.$$

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- Rearranging, we have

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- Substituting $D = 1/A$, we have

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Outline

Global redistribution networks

One more thing:

- How do we supply these facilities?
- How do we best redistribute mail? People?
- How do we get beer to the pubs?
- Gaster and Newman model: cost is a function of basic maintenance and travel time:

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- Travel time is more complicated: Take 'distance' between nodes to be a composite of shortest path distance ℓ_{ij} and number of legs to journey:

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- When $\delta = 1$, only number of hops matters.

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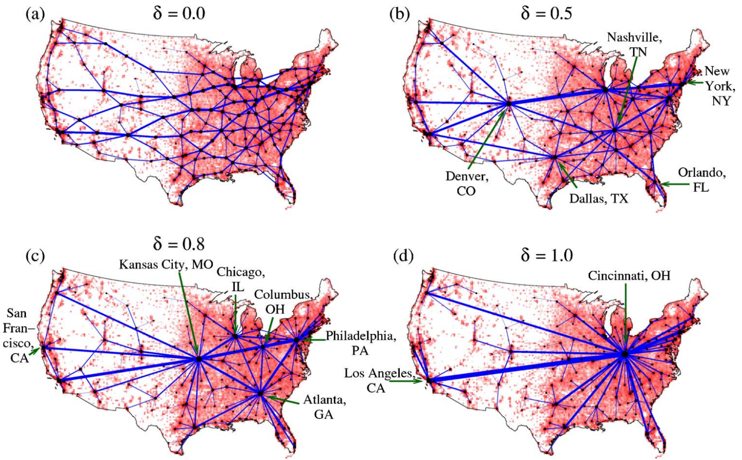

From Gastner and Newman (2006) [7]

Outline

Public versus private facilities

Beyond minimizing distances:

- "Scaling laws between population and facility densities" by Um et al., Proc. Natl. Acad. Sci., 2009. [23]
- Um et al. find empirically and argue theoretically that the connection between facility and population density

$$D \propto \rho^{\alpha}$$

  does not universally hold with $\alpha = 2/3$.
- Two idealized limiting classes:
  1. For-profit, commercial facilities: $\alpha = 1$;
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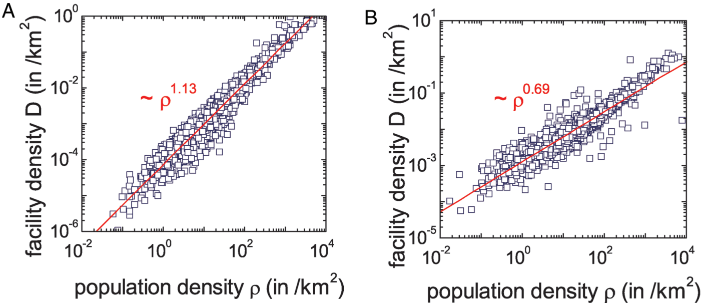
Public versus private facilities: evidence



- Left plot: ambulatory hospitals in the U.S.
- Right plot: public schools in the U.S.
- Note: break in scaling for public schools. Transition from $\alpha \simeq 2/3$ to $\alpha = 1$ around $\rho \simeq 100$.

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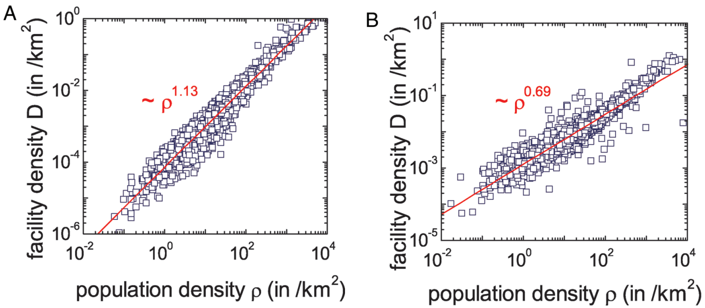

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Public versus private facilities: evidence

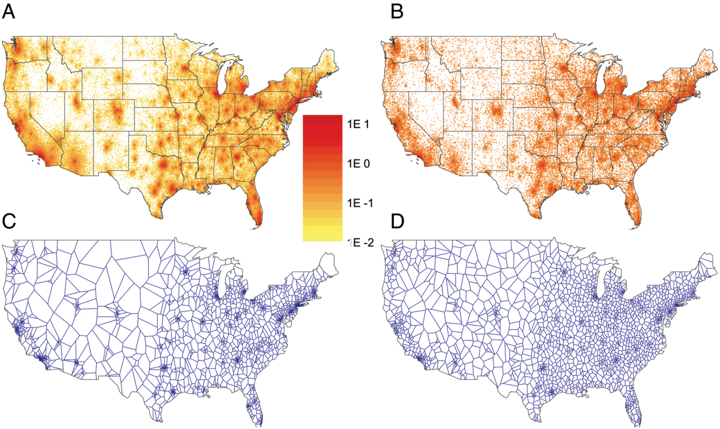
US facility	α (SE)	R^2
Ambulatory hospital	1.13(1)	0.93
Beauty care	1.08(1)	0.86
Laundry	1.05(1)	0.90
Automotive repair	0.99(1)	0.92
Private school	0.95(1)	0.82
Restaurant	0.93(1)	0.89
Accommodation	0.89(1)	0.70
Bank	0.88(1)	0.89
Gas station	0.86(1)	0.94
Death care	0.79(1)	0.80
* Fire station	0.78(3)	0.93
* Police station	0.71(6)	0.75
Public school	0.69(1)	0.87

SK facility	α (SE)	R^2
Bank	1.18(2)	0.96
Parking place	1.13(2)	0.91
* Primary clinic	1.09(2)	1.00
* Hospital	0.96(5)	0.97
* University/college	0.93(9)	0.89
Market place	0.87(2)	0.90
* Secondary school	0.77(3)	0.98
* Primary school	0.77(3)	0.97
Social welfare org.	0.75(2)	0.84
* Police station	0.71(5)	0.94
Government office	0.70(1)	0.93
* Fire station	0.60(4)	0.93
* Public health center	0.09(5)	0.19

Rough transition between public and private at $\alpha \simeq 0.8$.

Note: * indicates analysis is at state/province level; otherwise county level.

Public versus private facilities: evidence



A, C: ambulatory hospitals in the U.S.; B, D: public schools in the U.S.; A, B: data; C, D: Voronoi diagram from model simulation.

Public versus private facilities: the story

So what's going on?

- Social institutions seek to minimize distance of travel.
- Commercial institutions seek to maximize the number of visitors.
- Defns: For the ith facility and its Voronoi cell V_i, define
  - n_i = population of the ith cell;
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- Objective function to maximize for a facility (highly constructed):

$$v_i = n_i \langle r_i \rangle^{\beta} \text{ with } 0 \leq \beta \leq 1.$$

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References I

[1] J. R. Banavar, F. Colaiori, A. Flammini, A. Maritan, and A. Rinaldo.
Topology of the fittest transportation network.
Phys. Rev. Lett., 84:4745–4748, 2000. pdf (⊞)

[2] J. R. Banavar, A. Maritan, and A. Rinaldo.
Size and form in efficient transportation networks.
Nature, 399:130–132, 1999. pdf (⊞)

[3] S. Bohn and M. O. Magnasco.
Structure, scaling, and phase transition in the optimal transport network.
Phys. Rev. Lett., 98:088702, 2007. pdf (⊞)

[4] P. S. Dodds.
Optimal form of branching supply and collection networks.
Phys. Rev. Lett., 104(4):048702, Jan 2010. pdf (⊞)

References II

[5] P. S. Dodds and D. H. Rothman.
Geometry of river networks. I. Scaling, fluctuations, and deviations.
Physical Review E, 63(1):016115, 2001. pdf (⊞)

[6] M. T. Gastner and M. E. J. Newman.
Diffusion-based method for producing density-equalizing maps.
Proc. Natl. Acad. Sci., 101:7499–7504, 2004. pdf (⊞)

[7] M. T. Gastner and M. E. J. Newman.
Optimal design of spatial distribution networks.
Phys. Rev. E, 74:016117, 2006. pdf (⊞)

[8] M. T. Gastner and M. E. J. Newman.
Shape and efficiency in spatial distribution networks.
J. Stat. Mech.: Theor. & Exp., 1:P01015, 2006.
pdf (⊞)

References III

[9] S. M. Gusein-Zade.
Bunge's problem in central place theory and its generalizations.
Geogr. Anal., 14:246–252, 1982.

[10] K. A. McCulloh, J. S. Sperry, and F. R. Adler.
Water transport in plants obeys Murray's law.
Nature, 421:939–942, 2003. pdf (⊞)

[11] K. A. McCulloh, J. S. Sperry, and F. R. Adler.
Murray's law and the hydraulic vs mechanical functioning of wood.
Functional Ecology, 18:931–938, 2004. pdf (⊞)

[12] C. D. Murray.
The physiological principle of minimum work applied to the angle of branching of arteries.
J. Gen. Physiol., 9(9):835–841, 1926. pdf (⊞)

References IV

[13] C. D. Murray.
The physiological principle of minimum work. I. The vascular system and the cost of blood volume.
Proc. Natl. Acad. Sci., 12:207–214, 1926. pdf (⊞)

[14] C. D. Murray.
A relationship between circumference and weight in trees and its bearing on branching angles.
J. Gen. Physiol., 10:725–729, 1927. pdf (⊞)

[15] I. Rodríguez-Iturbe and A. Rinaldo.
Fractal River Basins: Chance and Self-Organization.
Cambridge University Press, Cambrigde, UK, 1997.

[16] A. E. Scheidegger.
Theoretical Geomorphology.
Springer-Verlag, New York, third edition, 1991.

References V

[17] W. R. Stahl.
Scaling of respiratory variables in mammals.
Journal of Applied Physiology, 22:453–460, 1967.

[18] G. E. Stephan.
Territorial division: The least-time constraint behind the formation of subnational boundaries.
Science, 196:523–524, 1977. pdf (⊞)

[19] G. E. Stephan.
Territorial subdivision.
Social Forces, 63:145–159, 1984. pdf (⊞)

[20] D. W. Thompson.
On Growth and From.
Cambridge University Pres, Great Britain, 2nd edition, 1952.

References VI

[21] D. W. Thompson.
On Growth and Form — Abridged Edition.
Cambridge University Press, Great Britain, 1961.

[22] D. L. Turcotte, J. D. Pelletier, and W. I. Newman.
Networks with side branching in biology.
Journal of Theoretical Biology, 193:577–592, 1998.

[23] J. Um, S.-W. Son, S.-I. Lee, H. Jeong, and B. J. Kim.
Scaling laws between population and facility densities.
Proc. Natl. Acad. Sci., 106:14236–14240, 2009.
pdf (⊞)

References VII

[24] P. D. Weinberg and C. R. Ethier.
Twenty-fold difference in hemodynamic wall shear stress between murine and human aortas.
Journal of Biomechanics, 40(7):1594–1598, 2007. pdf (⊞)

[25] G. B. West, J. H. Brown, and B. J. Enquist.
A general model for the origin of allometric scaling laws in biology.
Science, 276:122–126, 1997. pdf (⊞)

[26] Q. Xia.
The formation of a tree leaf.
Submitted. pdf (⊞)

References VIII

[27] Q. Xia.
Optimal paths related to transport problems.
Communications in Contemporary Mathematics, 5:251–279, 2003. pdf (⊞)