# Applications of Random Networks

## Complex Networks, CSYS/MATH 303, Spring, 2010

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont









Analysis of real networks
How to build revisited
Motifs
References

# Outline


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- How to build revisited
- Motifs

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# More on building random networks

- Problem: How much of a real network's structure is non-random?
- Key elephant in the room: the degree distribution $P_k$.
- First observe departure of $P_k$ from a Poisson distribution.
- Next: measure the departure of a real network with a degree frequency $N_k$ from a random network with the same degree frequency.
- Degree frequency $N_k$ = observed frequency of degrees for a real network.
- What we now need to do: Create an ensemble of random networks with degree frequency $N_k$ and then compare.

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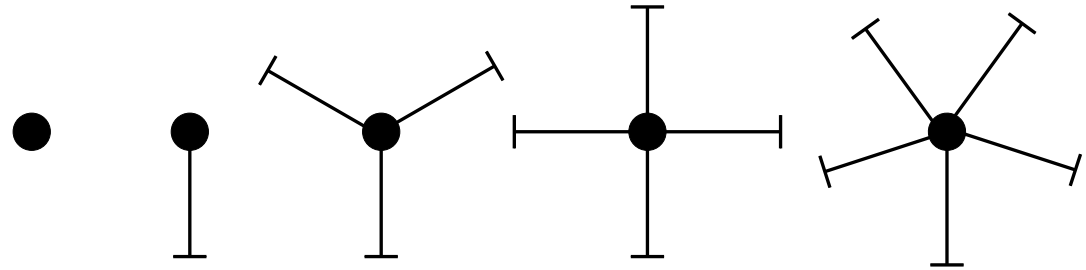
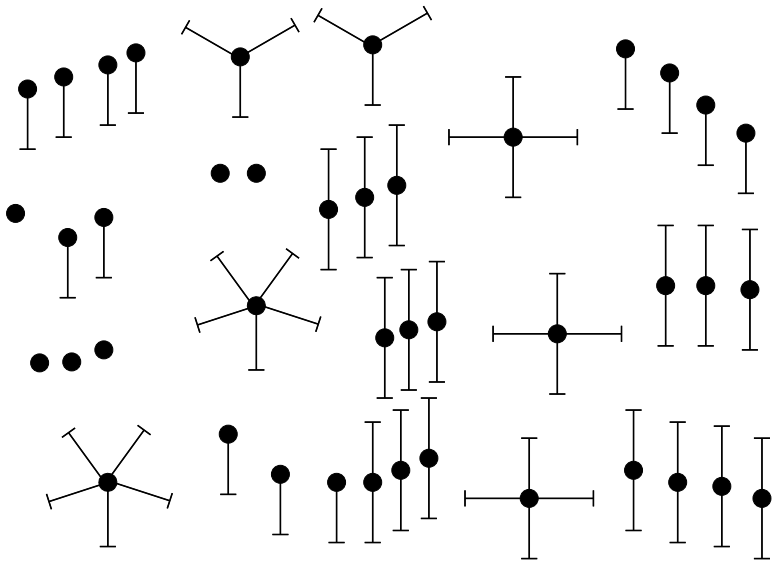
- Idea: start with a soup of unconnected nodes with stubs (half-edges):





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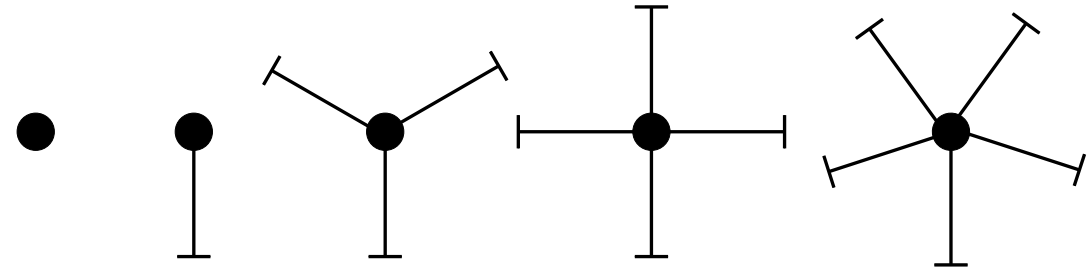

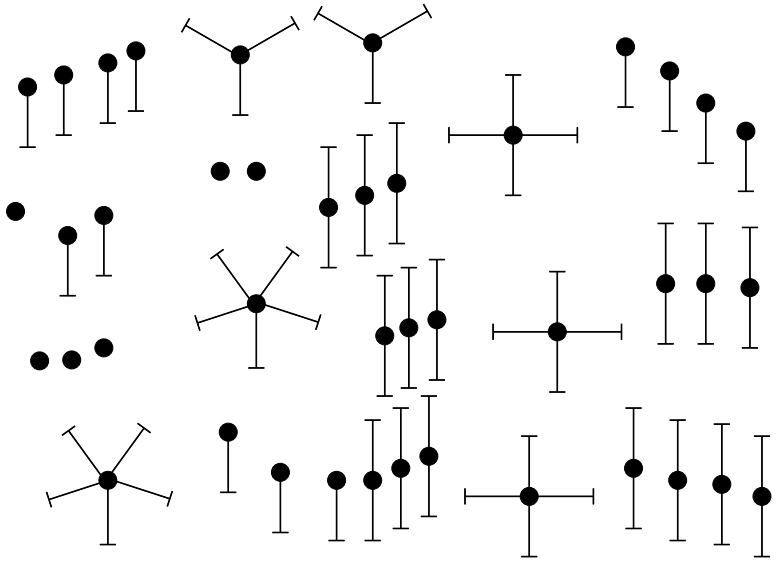

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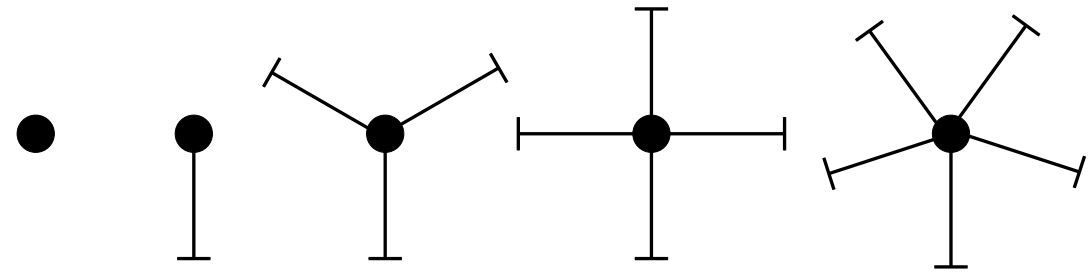

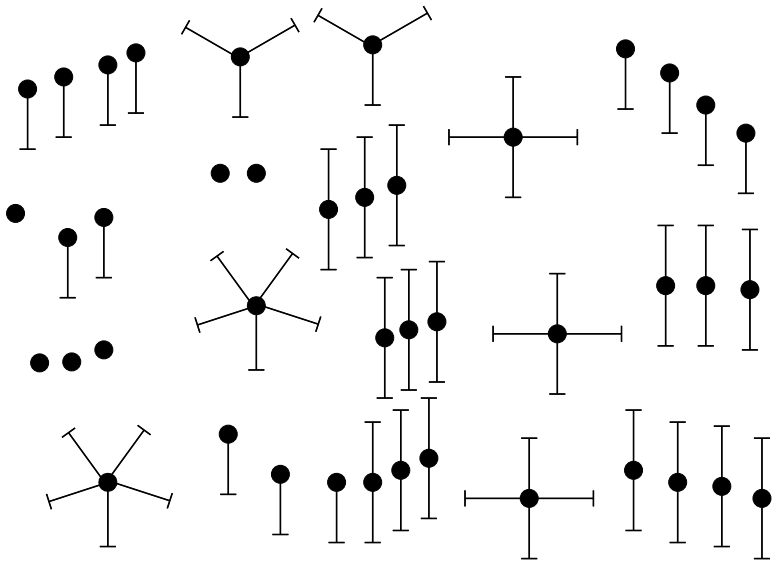

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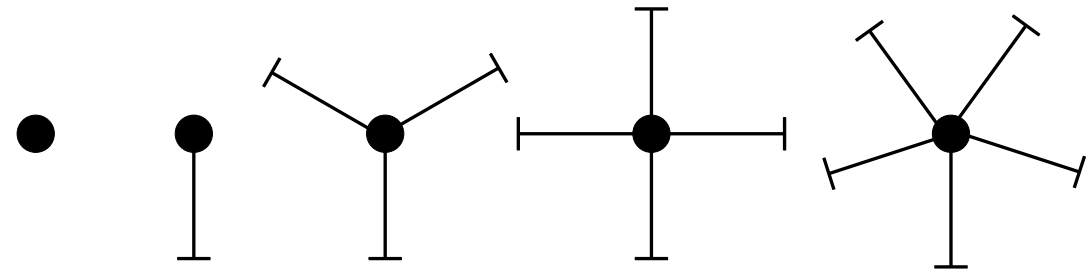

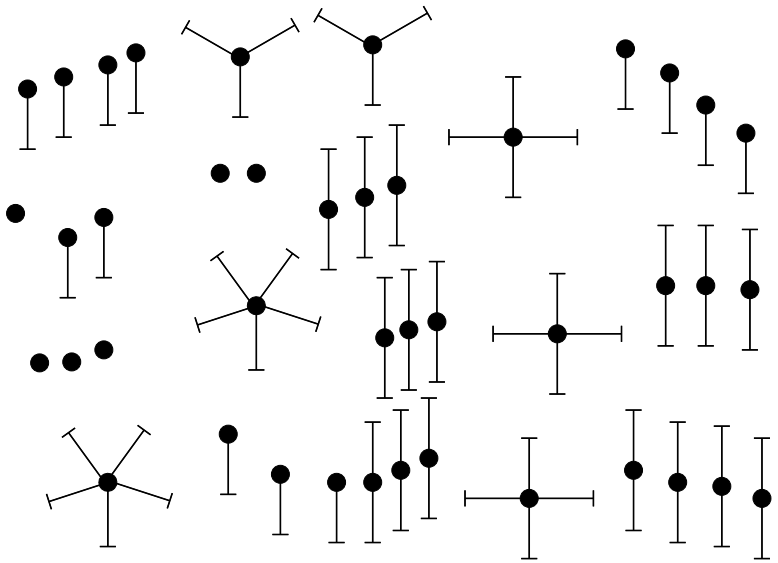

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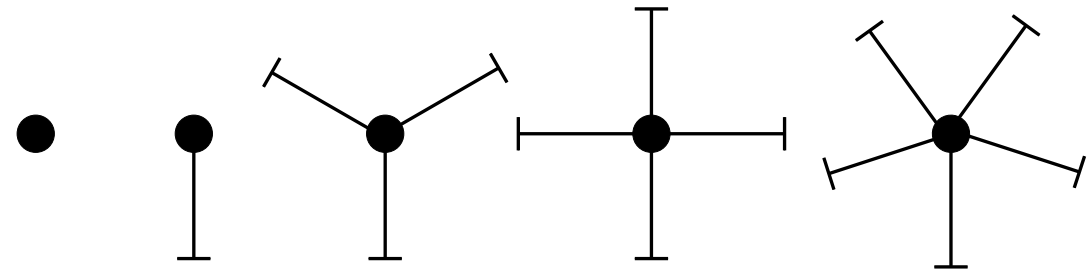

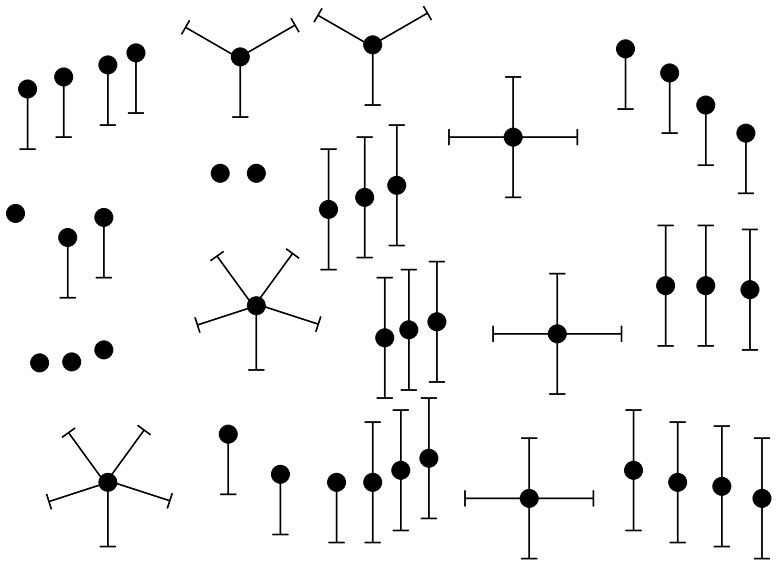

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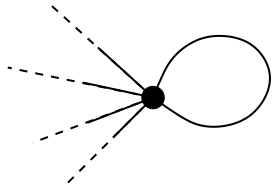
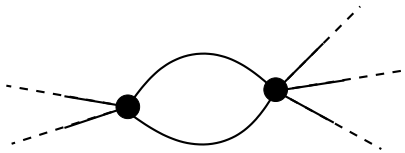
(A)  (B) 

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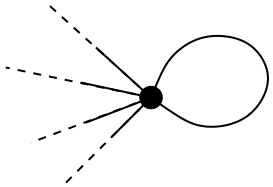
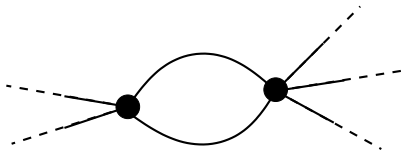
(A)  (B) 

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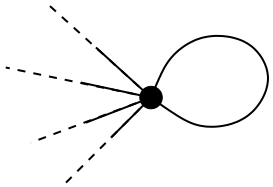
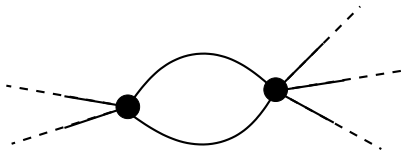
(A)  (B) 

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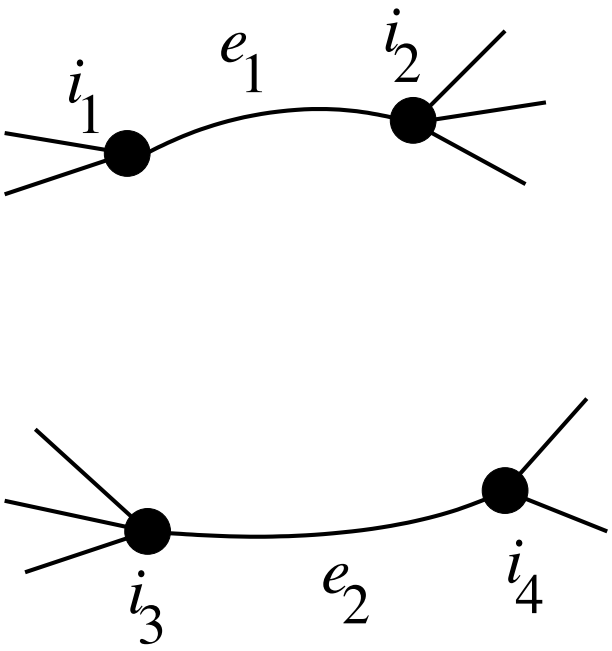


- Randomly choose two edges. (Or choose problem edge and a random edge)
- Check to make sure edges are disjoint.
- Rewire one end of each edge.
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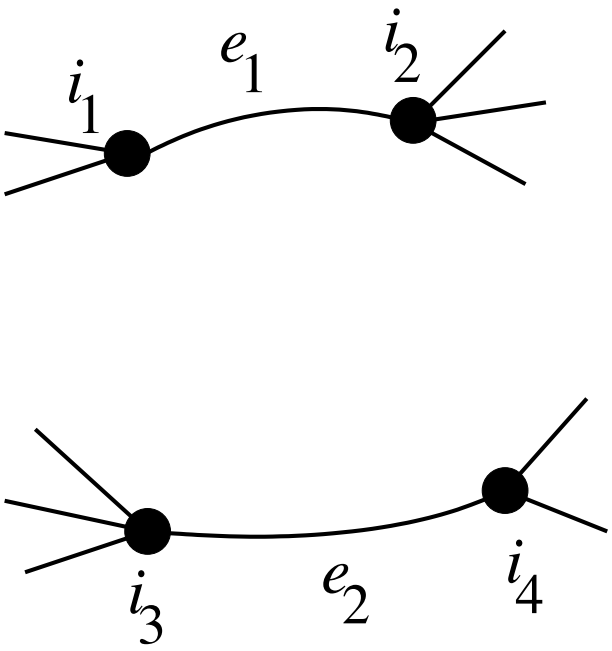


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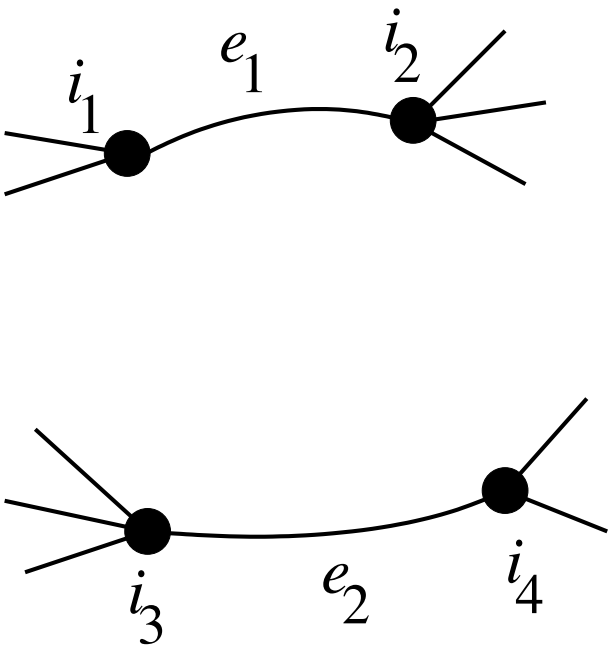


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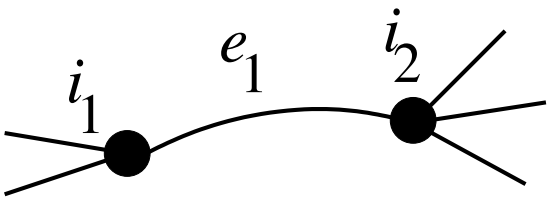


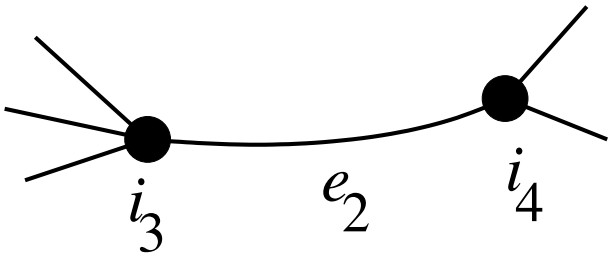


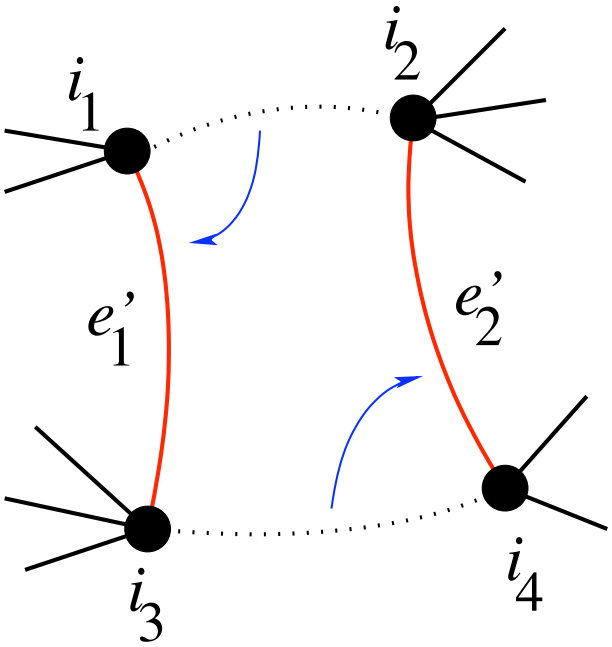


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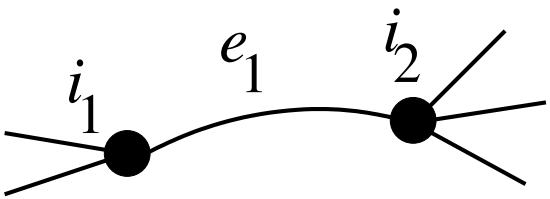


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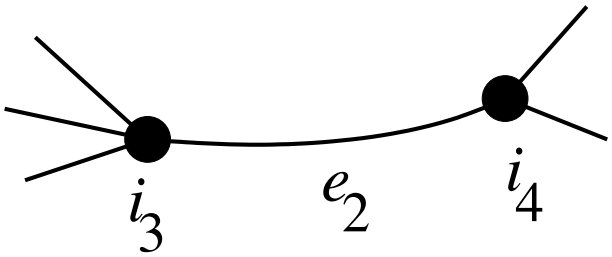


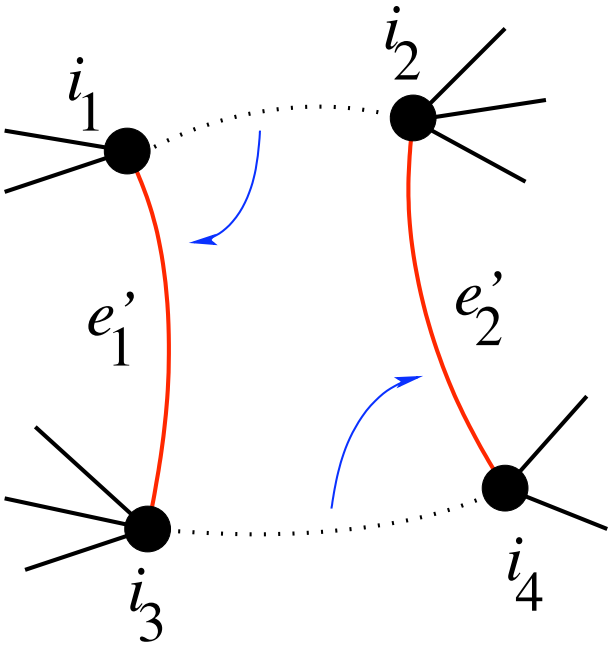


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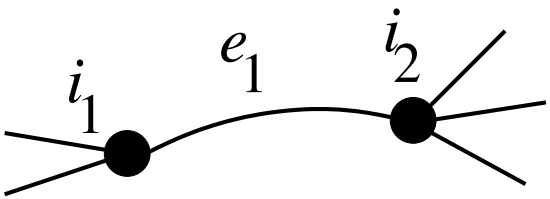


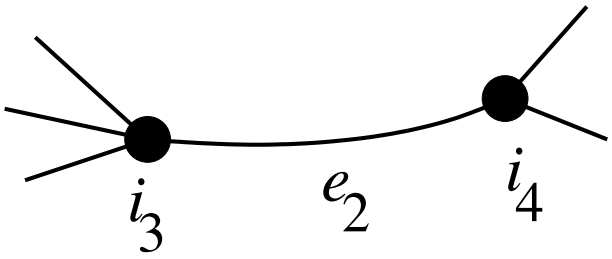


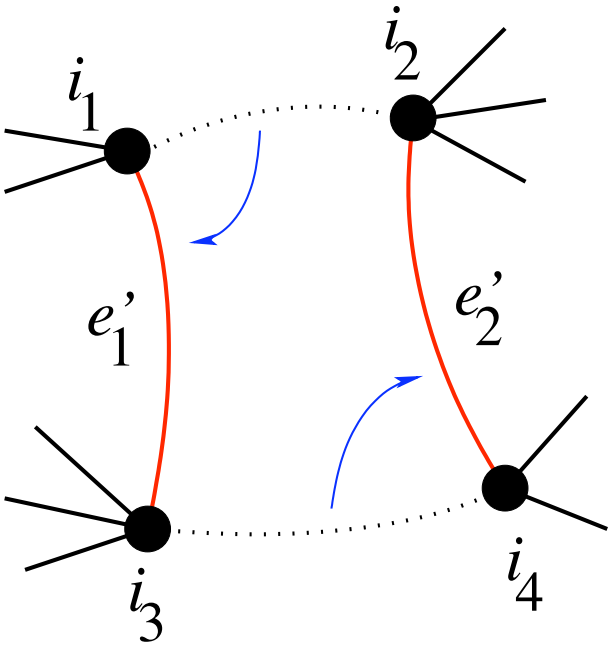


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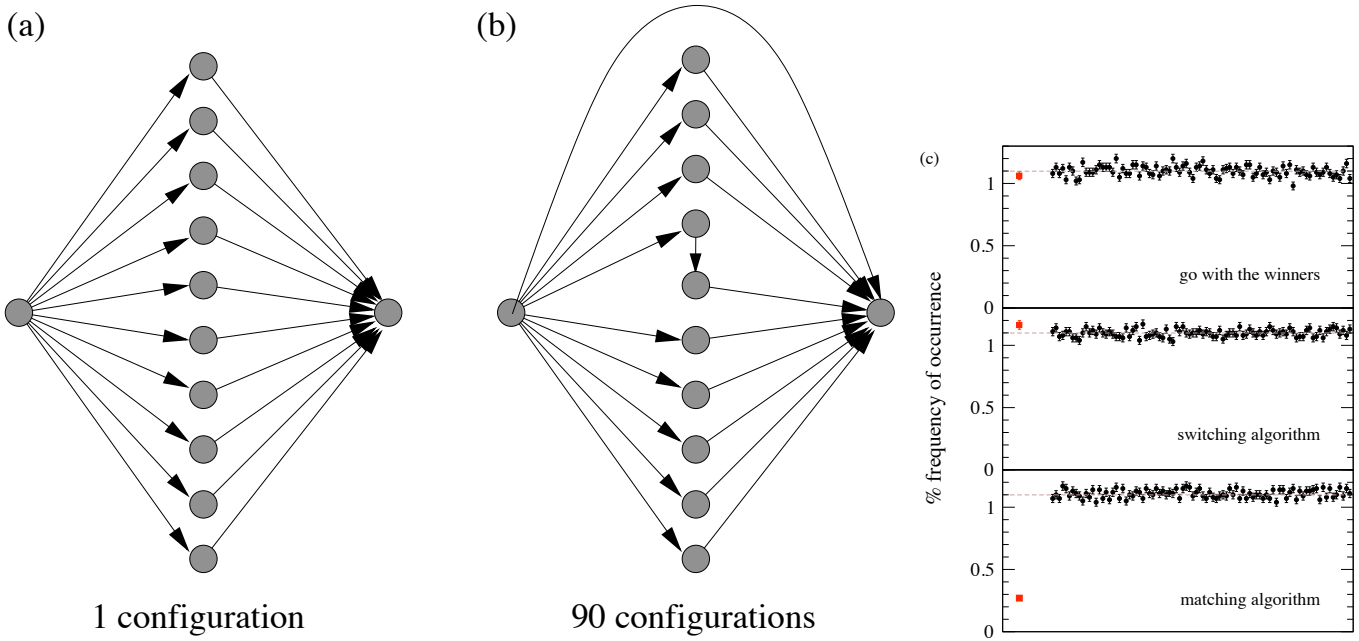

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- What if we have $P_k$ instead of $N_k$?
- Must now create nodes before start of the construction algorithm.
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- Looked at gene expression within full context of transcriptional regulation networks.
- Specific example of Escherichia coli.
- Directed network with 577 interactions (edges) and 424 operons (nodes).
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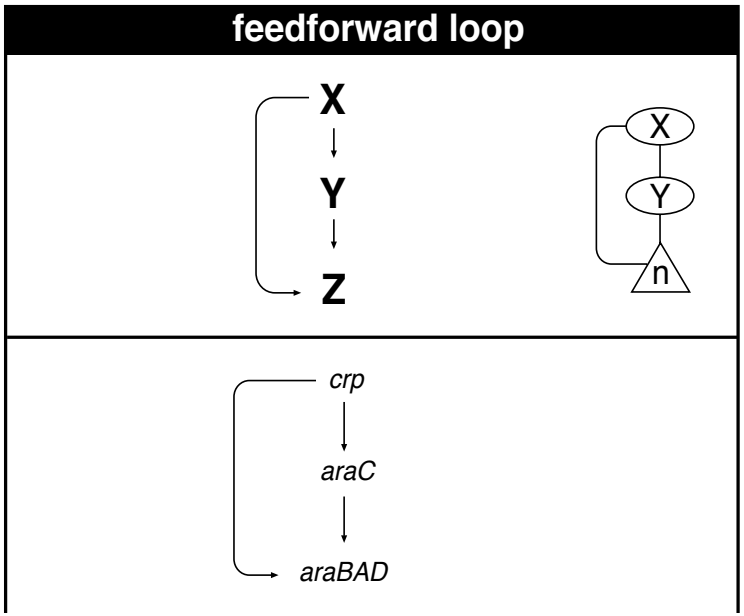


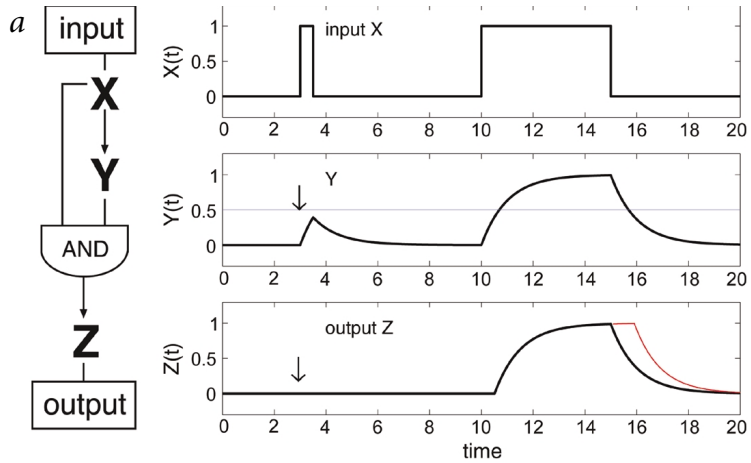


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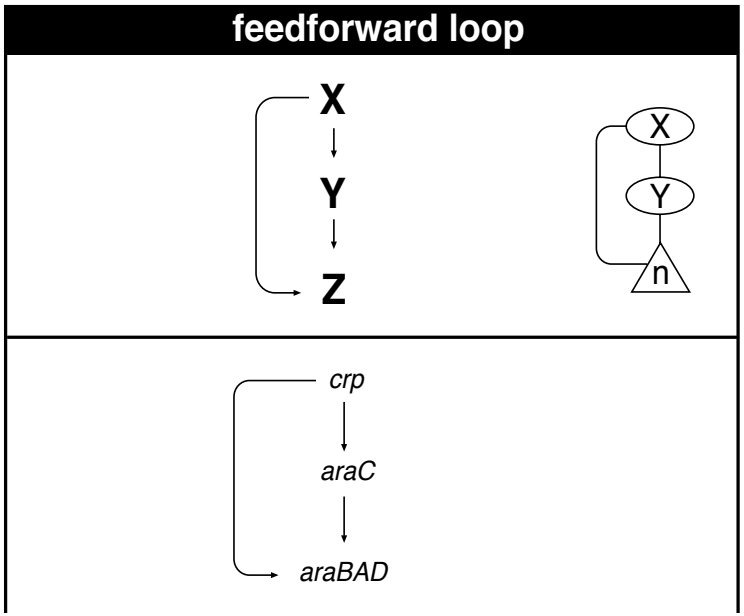


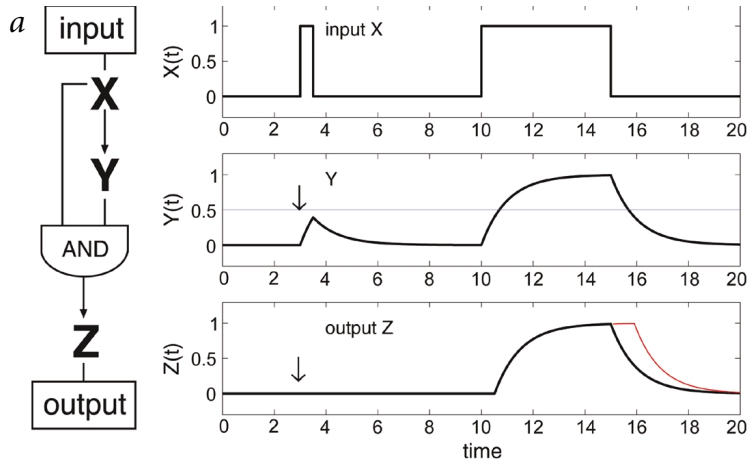


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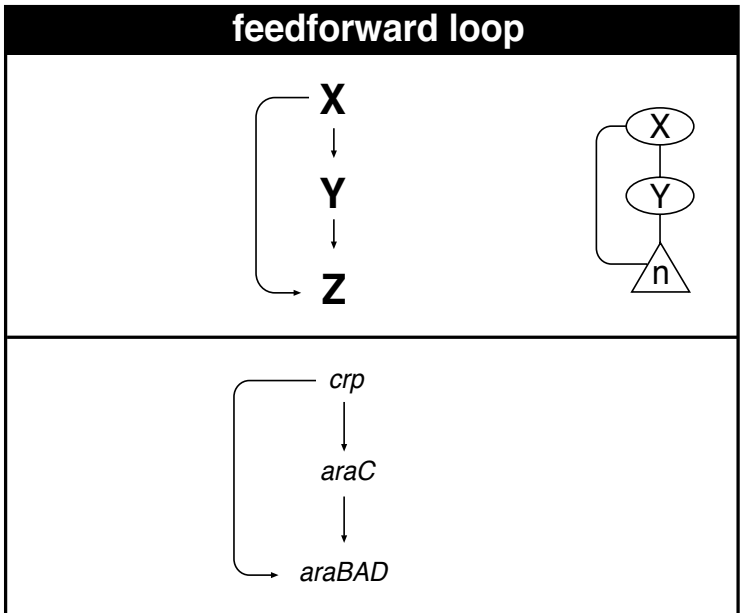
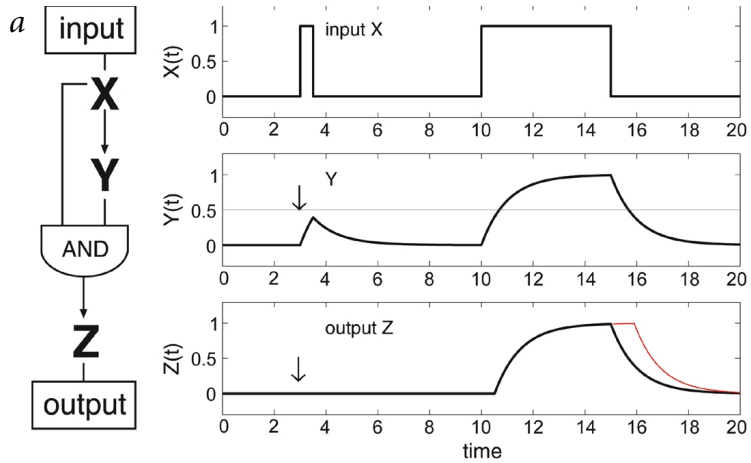
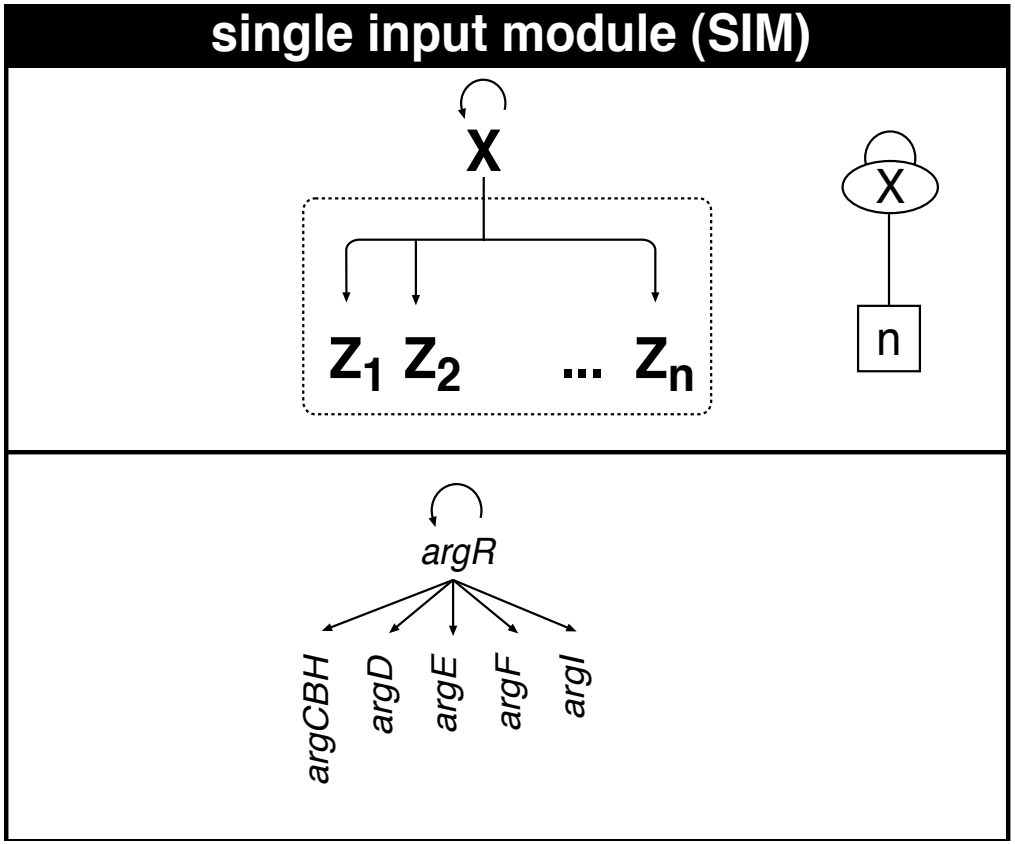


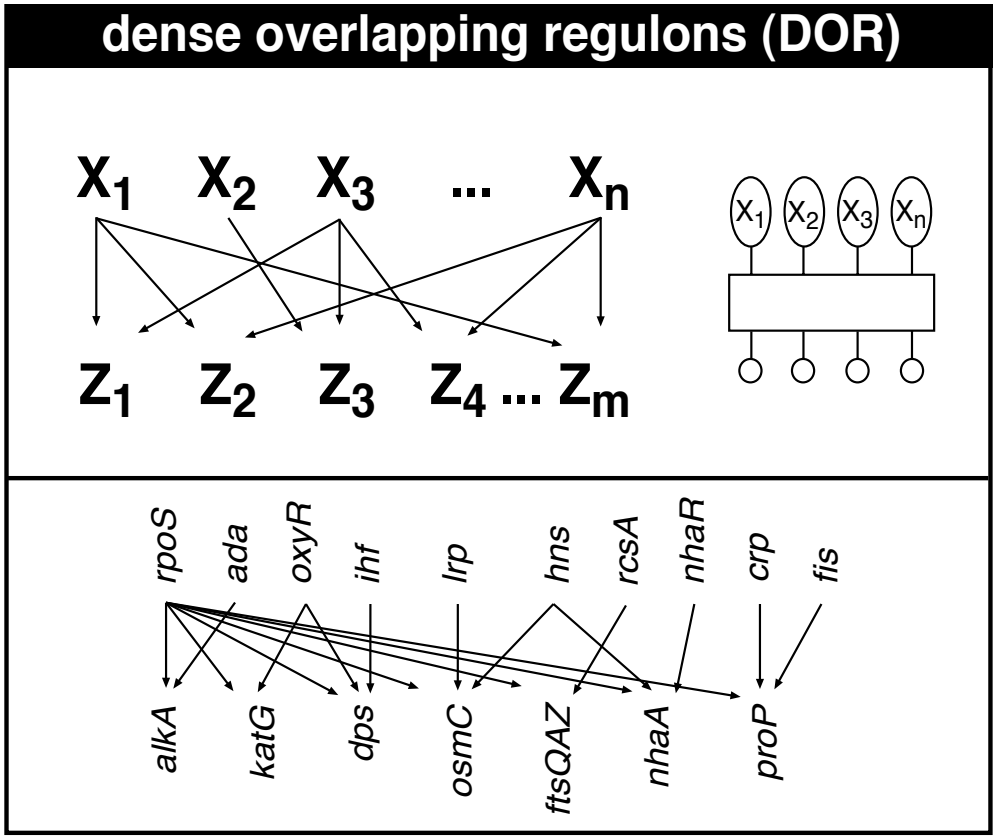

- $Z$ only turns on in response to sustained activity in $X$.
- Turning off $X$ rapidly turns off $Z$.
- Analogy to elevator doors.

# Network motifs




- Master switch.

# Network motifs



# Network motifs

- Note: selection of motifs to test is reasonable but nevertheless ad-hoc.
- For more, see work carried out by Wiggins et al. at Columbia.

# Network motifs

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# References I


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