

Random Networks

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- How to build
- Some visual examples

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- Degree distributions
- Configuration model
- Largest component

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Pure, abstract random networks:

- ▶ Consider set of all networks with N labelled nodes and m edges.
- ▶ Standard random network = **randomly chosen** network from this set.
- ▶ To be clear: each network is **equally** probable.
- ▶ Sometimes equiprobability is a good assumption, but it is always an assumption.
- ▶ Known as Erdős-Rényi random networks or **ER graphs**.

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Some features:

- ▶ Number of possible edges:

$$0 \leq m \leq \binom{N}{2} = \frac{N(N-1)}{2}$$

- ▶ Given m edges, there are $\binom{N}{m}$ different possible networks.
- ▶ Crazy factorial explosion for $1 \ll m \ll \binom{N}{2}$.
- ▶ Limit of $m = 0$: empty graph.
- ▶ Limit of $m = \binom{N}{2}$: complete or fully-connected graph.
- ▶ **Real world**: links are usually costly so real networks are almost always **sparse**.

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How to build standard random networks:

- ▶ Given N and m .
 - ▶ Two probabilistic methods (we'll see a third later on)
1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability p .
 - ▶ **Useful for theoretical work.**
 2. Take N nodes and add exactly m links by selecting edges without replacement.
 - ▶ **Algorithm:** Randomly choose a pair of nodes i and j , $i \neq j$, and connect if unconnected; repeat until all m edges are allocated.
 - ▶ Best for adding relatively small numbers of links (most cases).
 - ▶ 1 and 2 are effectively equivalent for large N .

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A few more things:

- ▶ For method 1, # links is probabilistic:

$$\langle m \rangle = p \binom{N}{2} = p \frac{1}{2} N(N-1)$$

- ▶ So the expected or **average degree** is

$$\begin{aligned} \langle k \rangle &= \frac{2 \langle m \rangle}{N} \\ &= \frac{2}{N} p \frac{1}{2} N(N-1) = \frac{2}{N} p \frac{1}{2} N(N-1) = p(N-1). \end{aligned}$$

- ▶ Which is what it should be...
- ▶ If we keep $\langle k \rangle$ constant then $p \propto 1/N \rightarrow 0$ as $N \rightarrow \infty$.

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Next slides:

Example realizations of random networks

- ▶ $N = 500$
- ▶ Vary m , the number of edges from 100 to 1000.
- ▶ Average degree $\langle k \rangle$ runs from 0.4 to 4.
- ▶ Look at full network plus the largest component.

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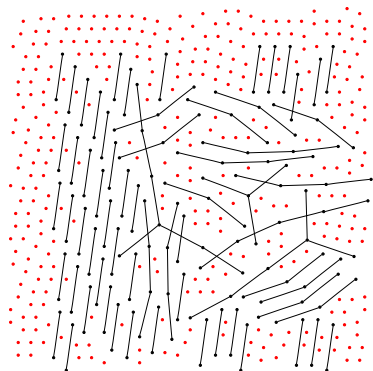
Size of the Giant Component

Average Component Size

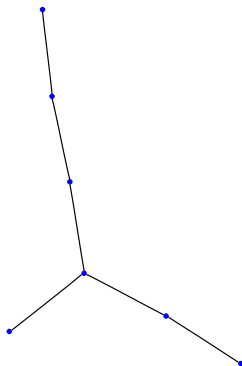
References

Random networks: examples

entire network:



largest component:



$N = 500$, number of edges $m = 100$
average degree $\langle k \rangle = 0.4$

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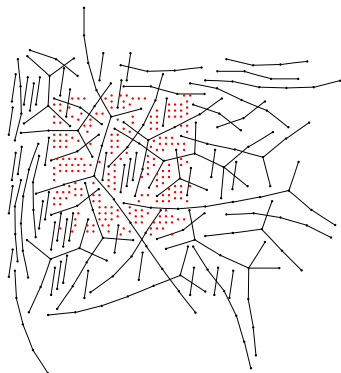
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Average Component Size

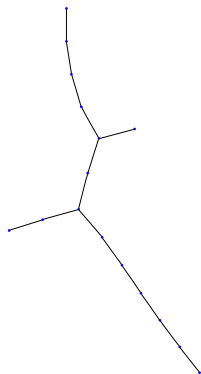
References

Random networks: examples

entire network:



largest component:



$N = 500$, number of edges $m = 200$
average degree $\langle k \rangle = 0.8$

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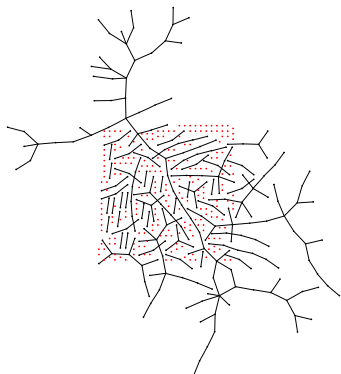
Size of the Giant Component

Average Component Size

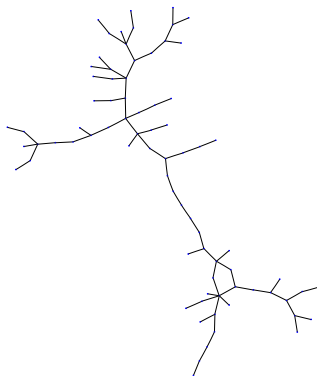
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entire network:



largest component:



$N = 500$, number of edges $m = 230$
average degree $\langle k \rangle = 0.92$

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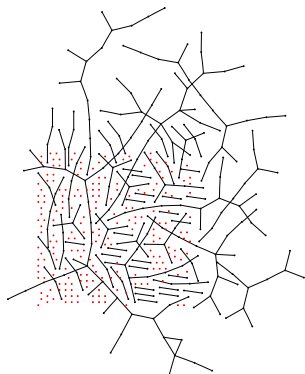
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Average Component Size

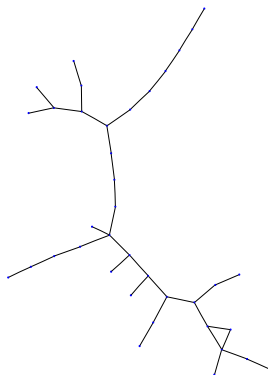
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largest component:



$N = 500$, number of edges $m = 240$
average degree $\langle k \rangle = 0.96$

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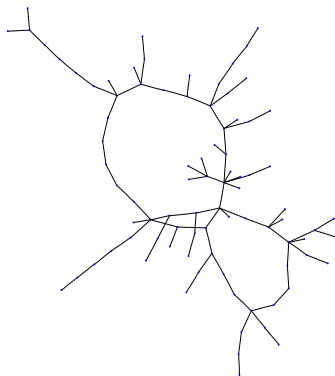
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largest component:



$N = 500$, number of edges $m = 250$
average degree $\langle k \rangle = 1$

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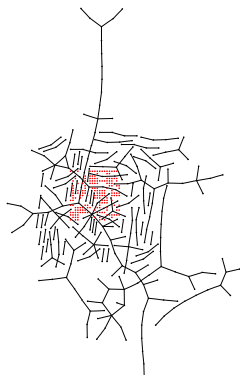
Size of the Giant Component

Average Component Size

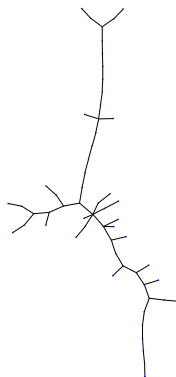
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largest component:



$N = 500$, number of edges $m = 260$
average degree $\langle k \rangle = 1.04$

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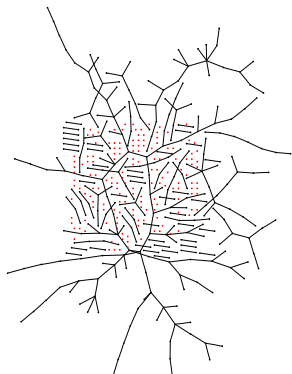
Size of the Giant Component

Average Component Size

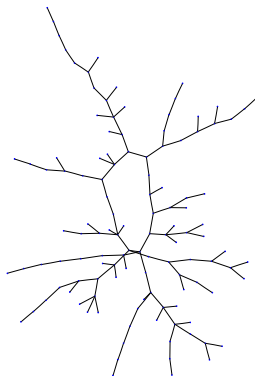
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largest component:



$N = 500$, number of edges $m = 280$
average degree $\langle k \rangle = 1.12$

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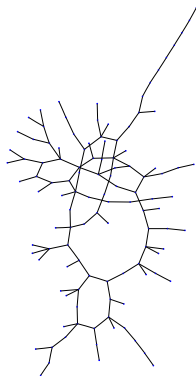
References

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largest component:



$N = 500$, number of edges $m = 300$
average degree $\langle k \rangle = 1.2$

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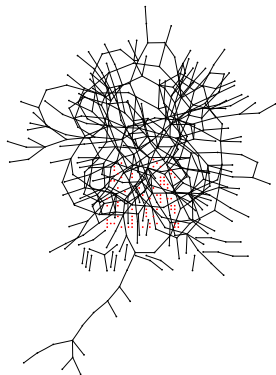
Size of the Giant Component

Average Component Size

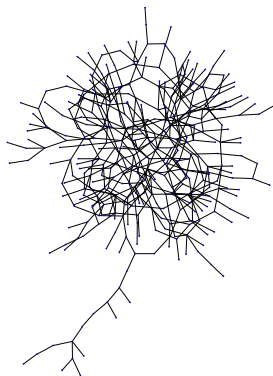
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largest component:



$N = 500$, number of edges $m = 500$
average degree $\langle k \rangle = 2$

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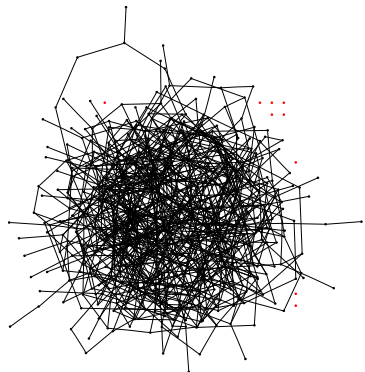
Size of the Giant Component

Average Component Size

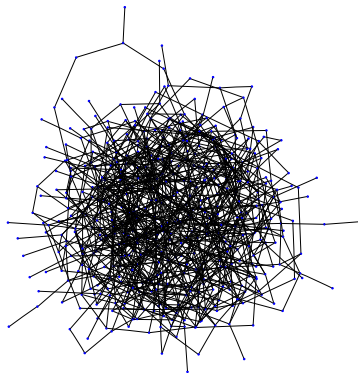
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$N = 500$, number of edges $m = 1000$
average degree $\langle k \rangle = 4$

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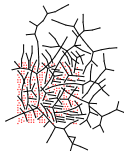
$m = 100$
 $\langle k \rangle = 0.4$



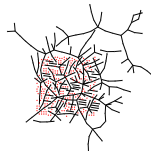
$m = 200$
 $\langle k \rangle = 0.8$



$m = 230$
 $\langle k \rangle = 0.92$



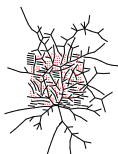
$m = 240$
 $\langle k \rangle = 0.96$



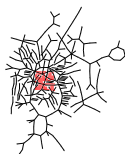
$m = 250$
 $\langle k \rangle = 1$



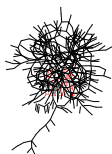
$m = 260$
 $\langle k \rangle = 1.04$



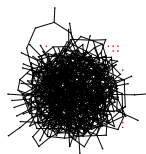
$m = 280$
 $\langle k \rangle = 1.12$



$m = 300$
 $\langle k \rangle = 1.2$



$m = 500$
 $\langle k \rangle = 2$



$m = 1000$
 $\langle k \rangle = 4$

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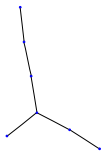
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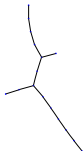
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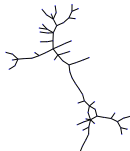
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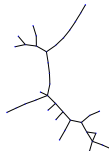
$m = 100$
 $\langle k \rangle = 0.4$



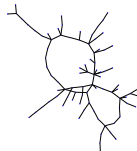
$m = 200$
 $\langle k \rangle = 0.8$



$m = 230$
 $\langle k \rangle = 0.92$



$m = 240$
 $\langle k \rangle = 0.96$



$m = 250$
 $\langle k \rangle = 1$



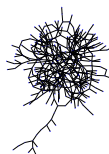
$m = 260$
 $\langle k \rangle = 1.04$



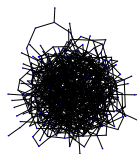
$m = 280$
 $\langle k \rangle = 1.12$



$m = 300$
 $\langle k \rangle = 1.2$



$m = 500$
 $\langle k \rangle = 2$



$m = 1000$
 $\langle k \rangle = 4$

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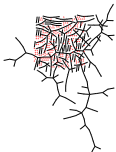
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$m = 250$
 $\langle k \rangle = 1$



$m = 250$
 $\langle k \rangle = 1$



$m = 250$
 $\langle k \rangle = 1$



$m = 250$
 $\langle k \rangle = 1$



$m = 250$
 $\langle k \rangle = 1$



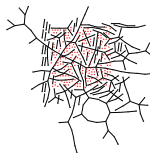
$m = 250$
 $\langle k \rangle = 1$



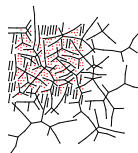
$m = 250$
 $\langle k \rangle = 1$



$m = 250$
 $\langle k \rangle = 1$



$m = 250$
 $\langle k \rangle = 1$



$m = 250$
 $\langle k \rangle = 1$

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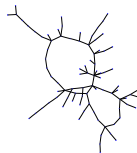
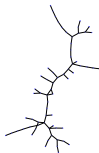
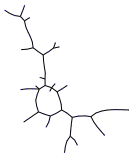
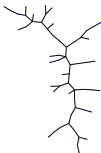
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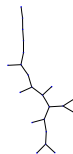
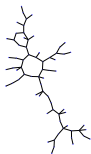
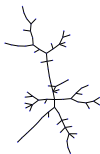
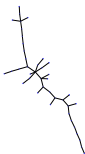
$m = 250$
 $\langle k \rangle = 1$

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Clustering:

- ▶ For method 1, what is the clustering coefficient for a finite network?
- ▶ Consider triangle/triple clustering coefficient (Newman^[1]):

$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$

- ▶ Recall: C_2 = probability that two nodes are connected given they have a friend in common.
- ▶ For standard random networks, we have simply that

$$C_2 = p.$$

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Clustering:

- ▶ So for large random networks ($N \rightarrow \infty$), clustering drops to zero.
- ▶ Key structural feature of random networks is that they locally look like branching networks (no loops).

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Degree distribution:

- ▶ Recall p_k = probability that a randomly selected node has degree k .
- ▶ Consider method 1 for constructing random networks: each possible link is realized with probability p .
- ▶ Now consider one node: there are ' $N - 1$ choose k ' ways the node can be connected to k of the other $N - 1$ nodes.
- ▶ Each connection occurs with probability p , each non-connection with probability $(1 - p)$.
- ▶ Therefore have a binomial distribution:

$$P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

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Limiting form of $P(k; p, N)$:

- ▶ Our degree distribution:

$$P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

- ▶ What happens as $N \rightarrow \infty$?
- ▶ We must end up with the normal distribution right?
- ▶ If p is fixed, then we would end up with a Gaussian with average degree $\langle k \rangle \simeq pN \rightarrow \infty$.
- ▶ But we want to keep $\langle k \rangle$ fixed...
- ▶ So examine limit of $P(k; p, N)$ when $p \rightarrow 0$ and $N \rightarrow \infty$ with $\langle k \rangle = p(N-1) = \text{constant}$.

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Limiting form of $P(k; p, N)$:

- Substitute $p = \frac{\langle k \rangle}{N-1}$ into $P(k; p, N)$ and hold k fixed:

$$\begin{aligned} P(k; p, N) &= \binom{N-1}{k} \left(\frac{\langle k \rangle}{N-1} \right)^k \left(1 - \frac{\langle k \rangle}{N-1} \right)^{N-1-k} \\ &= \frac{(N-1)!}{k!(N-1-k)!} \frac{\langle k \rangle^k}{(N-1)^k} \left(1 - \frac{\langle k \rangle}{N-1} \right)^{N-1-k} \\ &= \frac{(N-1)(N-2)\cdots(N-k)}{k!} \frac{\langle k \rangle^k}{(N-1)^k} \left(1 - \frac{\langle k \rangle}{N-1} \right)^{N-1-k} \\ &\approx \frac{\cancel{N^k} \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{k}{N}\right)}{k! \cancel{N^k}} \frac{\langle k \rangle^k}{\left(1 - \frac{1}{N}\right)^k} \left(1 - \frac{\langle k \rangle}{N-1} \right)^{N-1-k} \end{aligned}$$

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Limiting form of $P(k; p, N)$:

- ▶ We are now here:

$$P(k; p, N) \simeq \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N-1} \right)^{N-1-k}$$

- ▶ Now use the excellent result:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n} \right)^n = e^x.$$

(Use l'Hôpital's rule to prove.)

- ▶ Identifying $n = N - 1$ and $x = -\langle k \rangle$:

$$P(k; \langle k \rangle) \simeq \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \left(1 - \frac{\langle k \rangle}{N-1} \right)^{-k} \rightarrow \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$$

- ▶ This is a Poisson distribution (⊕) with mean $\langle k \rangle$.

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General random networks

- ▶ So... standard random networks have a Poisson degree distribution
- ▶ Generalize to arbitrary degree distribution P_k .
- ▶ Also known as the **configuration model**^[1].
- ▶ Can generalize construction method from ER random networks.
- ▶ Assign each node a weight w from some distribution P_w and form links with probability

$$P(\text{link between } i \text{ and } j) \propto w_i w_j.$$

- ▶ But we'll be more interested in
 1. Randomly wiring up (and rewiring) already existing nodes with fixed degrees.
 2. Examining mechanisms that lead to networks with certain degree distributions.

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Coming up:

Example realizations of random networks with power law degree distributions:

- ▶ $N = 1000$.
- ▶ $P_k \propto k^{-\gamma}$ for $k \geq 1$.
- ▶ Set $P_0 = 0$ (no isolated nodes).
- ▶ Vary exponent γ between 2.10 and 2.91.
- ▶ Again, look at full network plus the largest component.
- ▶ Apart from degree distribution, wiring is random.

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Random networks: examples for $N=1000$

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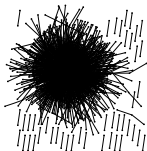
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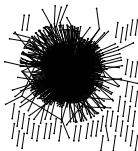
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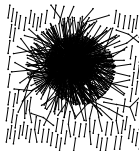
$\gamma = 2.1$
 $\langle k \rangle = 3.448$



$\gamma = 2.19$
 $\langle k \rangle = 2.986$



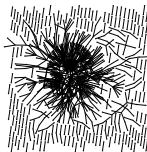
$\gamma = 2.28$
 $\langle k \rangle = 2.306$



$\gamma = 2.37$
 $\langle k \rangle = 2.504$



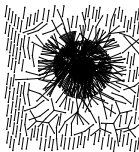
$\gamma = 2.46$
 $\langle k \rangle = 1.856$



$\gamma = 2.55$
 $\langle k \rangle = 1.712$



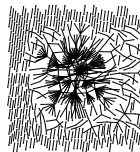
$\gamma = 2.64$
 $\langle k \rangle = 1.6$



$\gamma = 2.73$
 $\langle k \rangle = 1.862$



$\gamma = 2.82$
 $\langle k \rangle = 1.386$



$\gamma = 2.91$
 $\langle k \rangle = 1.49$

Random networks: largest components

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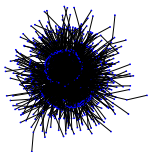
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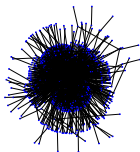
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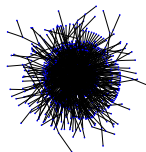
$\gamma = 2.1$
 $\langle k \rangle = 3.448$



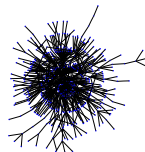
$\gamma = 2.19$
 $\langle k \rangle = 2.986$



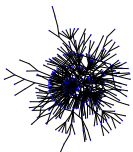
$\gamma = 2.28$
 $\langle k \rangle = 2.306$



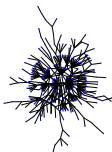
$\gamma = 2.37$
 $\langle k \rangle = 2.504$



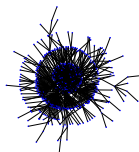
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 $\langle k \rangle = 1.856$



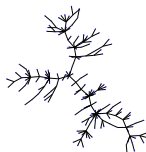
$\gamma = 2.55$
 $\langle k \rangle = 1.712$



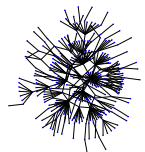
$\gamma = 2.64$
 $\langle k \rangle = 1.6$



$\gamma = 2.73$
 $\langle k \rangle = 1.862$



$\gamma = 2.82$
 $\langle k \rangle = 1.386$



$\gamma = 2.91$
 $\langle k \rangle = 1.49$

Poisson basics:

- ▶ Normalization: we must have

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1$$

- ▶ Checking:

$$\begin{aligned} \sum_{k=0}^{\infty} P(k; \langle k \rangle) &= \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \\ &= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} \\ &= e^{-\langle k \rangle} e^{\langle k \rangle} = 1 \checkmark \end{aligned}$$

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Poisson basics:

- ▶ Mean degree: we must have

$$\langle k \rangle = \sum_{k=0}^{\infty} kP(k; \langle k \rangle).$$

- ▶ Checking:

$$\begin{aligned} \sum_{k=0}^{\infty} kP(k; \langle k \rangle) &= \sum_{k=0}^{\infty} k \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \\ &= e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\langle k \rangle^k}{(k-1)!} \\ &= \langle k \rangle e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\langle k \rangle^{k-1}}{(k-1)!} \\ &= \langle k \rangle e^{-\langle k \rangle} \sum_{i=0}^{\infty} \frac{\langle k \rangle^i}{i!} = \langle k \rangle e^{-\langle k \rangle} e^{\langle k \rangle} = \langle k \rangle \checkmark \end{aligned}$$

- ▶ We'll get to a better way of doing this...

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Poisson basics:

- ▶ The **variance** of degree distributions for random networks turns out to be **very important**.
- ▶ Use calculation similar to one for finding $\langle k \rangle$ to find the **second moment**:

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

- ▶ Variance is then

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle^2 + \langle k \rangle - \langle k \rangle^2 = \langle k \rangle.$$

- ▶ So standard deviation σ is equal to $\sqrt{\langle k \rangle}$.
- ▶ Note: This is a special property of Poisson distribution and can trip us up...

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The edge-degree distribution:

- ▶ The degree distribution P_k is fundamental for our description of many complex networks
- ▶ Again: P_k is the degree of **randomly chosen node**.
- ▶ A second very important distribution arises from **choosing randomly on edges** rather than on nodes.
- ▶ Define Q_k to be the probability the node at a **random end** of a **randomly chosen edge** has degree k .
- ▶ Now choosing nodes based on their degree (i.e., size):

$$Q_k \propto kP_k$$

- ▶ Normalized form:

$$Q_k = \frac{kP_k}{\sum_{k'=0}^{\infty} k'P_{k'}} = \frac{kP_k}{\langle k \rangle}.$$

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The edge-degree distribution:

- ▶ For random networks, Q_k is also the probability that a friend (neighbor) of a random node has k friends.
- ▶ Useful variant on Q_k :

R_k = probability that a friend of a random node has k other friends.

$$R_k = \frac{(k+1)P_{k+1}}{\sum_{k'=0} (k'+1)P_{k'+1}} = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$

- ▶ Equivalent to friend having degree $k + 1$.
- ▶ **Natural question:** what's the expected number of other friends that one friend has?

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The edge-degree distribution:

- ▶ Given R_k is the probability that a friend has k other friends, then the average number of **friends' other friends** is

$$\begin{aligned}\langle k \rangle_R &= \sum_{k=0}^{\infty} k R_k = \sum_{k=0}^{\infty} k \frac{(k+1) P_{k+1}}{\langle k \rangle} \\ &= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k(k+1) P_{k+1} \\ &= \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} ((k+1)^2 - (k+1)) P_{k+1}\end{aligned}$$

(where we have sneakily matched up indices)

$$\begin{aligned}&= \frac{1}{\langle k \rangle} \sum_{j=0}^{\infty} (j^2 - j) P_j \quad (\text{using } j = k+1) \\ &= \frac{1}{\langle k \rangle} (\langle k^2 \rangle - \langle k \rangle)\end{aligned}$$

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The edge-degree distribution:

- ▶ Note: our result, $\langle k \rangle_R = \frac{1}{\langle k \rangle} (\langle k^2 \rangle - \langle k \rangle)$, is true for **all** random networks, **independent of degree distribution**.

- ▶ For standard random networks, recall

$$\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle.$$

- ▶ Therefore:

$$\langle k \rangle_R = \frac{1}{\langle k \rangle} (\langle k \rangle^2 + \langle k \rangle - \langle k \rangle) = \langle k \rangle$$

- ▶ Again, neatness of results is a special property of the Poisson distribution.
- ▶ So friends on average have $\langle k \rangle$ other friends, and $\langle k \rangle + 1$ total friends...

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Two reasons why this matters

Reason #1:

- ▶ Average # friends of friends per node is

$$\langle k_2 \rangle = \langle k \rangle \times \langle k \rangle_R = \langle k \rangle \frac{1}{\langle k \rangle} (\langle k^2 \rangle - \langle k \rangle) = \langle k^2 \rangle - \langle k \rangle.$$

- ▶ Key: Average depends on the **1st and 2nd moments** of P_k and **not just the 1st moment**.
- ▶ Three peculiarities:
 1. We might guess $\langle k_2 \rangle = \langle k \rangle (\langle k \rangle - 1)$ but it's actually $\langle k(k-1) \rangle$.
 2. If P_k has a **large second moment**, then $\langle k_2 \rangle$ will be big. (e.g., in the case of a power-law distribution)
 3. Your friends are different to you...

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Two reasons why this matters

More on peculiarity #3:

- ▶ A node's average # of friends: $\langle k \rangle$
- ▶ Friend's average # of friends: $\frac{\langle k^2 \rangle}{\langle k \rangle}$
- ▶ Comparison:

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle \frac{\langle k^2 \rangle}{\langle k \rangle^2} = \langle k \rangle \frac{\sigma^2 + \langle k \rangle^2}{\langle k \rangle^2} = \langle k \rangle \left(1 + \frac{\sigma^2}{\langle k \rangle^2} \right) \geq \langle k \rangle$$

- ▶ So only if everyone has the same degree (variance = $\sigma^2 = 0$) can a node be the same as its friends.
- ▶ Intuition: for random networks, the more connected a node, the more likely it is to be chosen as a friend.

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Two reasons why this matters

(Big) Reason #2:

- ▶ $\langle k \rangle_R$ is **key** to understanding how well random networks are connected together.
- ▶ e.g., we'd like to know what's the size of the largest component within a network.
- ▶ As $N \rightarrow \infty$, does our network have a **giant component**?
- ▶ **Defn:** Component = connected subnetwork of nodes such that \exists path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.
- ▶ **Defn:** Giant component = component that comprises a non-zero fraction of a network as $N \rightarrow \infty$.
- ▶ Note: Component = Cluster

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Giant component:

- ▶ A giant component exists if when we follow a random edge, we are likely to hit a node with **at least 1** other outgoing edge.
- ▶ Equivalently, expect exponential growth in node number as we move out from a random node.
- ▶ All of this is the same as requiring $\langle k \rangle_R > 1$.
- ▶ **Giant component condition** (or percolation condition):

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$$

- ▶ Again, see that the second moment is an essential part of the story.
- ▶ Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$

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Giant component

Standard random networks:

- ▶ Recall $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.
- ▶ Condition for giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle - \langle k \rangle}{\langle k \rangle} = \langle k \rangle$$

- ▶ Therefore when $\langle k \rangle > 1$, standard random networks have a giant component.
- ▶ When $\langle k \rangle < 1$, all components are finite.
- ▶ Fine example of a continuous phase transition (田).
- ▶ We say $\langle k \rangle = 1$ marks the critical point of the system.

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Random networks with skewed P_k :

- ▶ e.g, if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$ then

$$\begin{aligned}\langle k^2 \rangle &= c \sum_{k=0}^{\infty} k^2 k^{-\gamma} \\ &\sim \int_{x=0}^{\infty} x^{2-\gamma} dx \\ &\propto x^{3-\gamma} \Big|_{x=0}^{\infty} = \infty \quad (> \langle k \rangle).\end{aligned}$$

- ▶ So giant component **always exists** for these kinds of networks.
- ▶ Cutoff scaling is k^{-3} : if $\gamma > 3$ then we have to look harder at $\langle k \rangle_R$.

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And how big is the largest component?

- ▶ Define S_1 as the **size of the largest component**.
- ▶ Consider an infinite ER random network with average degree $\langle k \rangle$.
- ▶ Let's find S_1 with a back-of-the-envelope argument.
- ▶ Define δ as the probability that a randomly chosen node **does not** belong to the largest component.
- ▶ Simple connection: $\delta = 1 - S_1$.
- ▶ **Dirty trick**: If a randomly chosen node is not part of the largest component, then none of its neighbors are.
- ▶ So

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k$$

- ▶ Substitute in Poisson distribution...

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- ▶ Carrying on:

$$\begin{aligned}\delta &= \sum_{k=0}^{\infty} P_k \delta^k = \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle} \delta^k \\ &= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{(\langle k \rangle \delta)^k}{k!} \\ &= e^{-\langle k \rangle} e^{\langle k \rangle \delta} = e^{-\langle k \rangle (1-\delta)}.\end{aligned}$$

- ▶ Now substitute in $\delta = 1 - S_1$ and rearrange to obtain:

$$S_1 = 1 - e^{-\langle k \rangle S_1}.$$

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- ▶ We can figure out some limits and details for $S_1 = 1 - e^{-\langle k \rangle S_1}$.
- ▶ First, we can write $\langle k \rangle$ in terms of S_1 :

$$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}.$$

- ▶ As $\langle k \rangle \rightarrow 0$, $S_1 \rightarrow 0$.
- ▶ As $\langle k \rangle \rightarrow \infty$, $S_1 \rightarrow 1$.
- ▶ Notice that at $\langle k \rangle = 1$, the critical point, $S_1 = 0$.
- ▶ Only solvable for $S > 0$ when $\langle k \rangle > 1$.
- ▶ Really a transcritical bifurcation [2].

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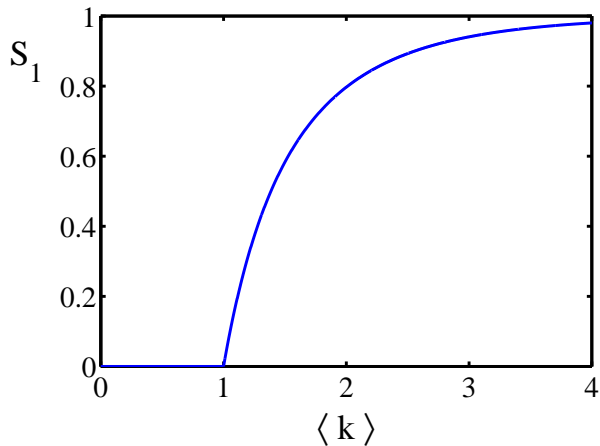
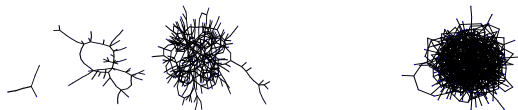
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Giant component

Turns out we were lucky...

- ▶ Our dirty trick **only works for** ER random networks.
- ▶ **The problem:** We assumed that neighbors have the same probability δ of belonging to the largest component.
- ▶ But we know our friends are different from us...
- ▶ Works for ER random networks because $\langle k \rangle = \langle k \rangle_R$.
- ▶ We need a separate probability δ' for the chance that a node **at the end of a random edge** is part of the largest component.
- ▶ We can do this but we need to enhance our toolkit with **Generatingfunctionology**... [3]
- ▶ (Well, not really but it's fun and we get all sorts of other things...)

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Generating functions

- ▶ **Idea:** Given a sequence a_0, a_1, a_2, \dots , associate each element with a distinct function or other mathematical object.
- ▶ Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

Definition:

- ▶ The **generating function** (g.f.) for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- ▶ Roughly: transforms a vector in R^∞ into a function defined on R^1 .
- ▶ Related to Fourier, Laplace, Mellin, ...

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Rolling dice:

- ▶ $p_k^{(\square)} = \Pr(\text{throwing a } k) = 1/6$ where $k = 1, 2, \dots, 6$.



$$F^{(\square)}(x) = \sum_{k=1}^6 p_k x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

- ▶ We'll come back to this simple example as we derive various delicious properties of generating functions.

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Example

- ▶ Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where $c = 1 - e^{-\lambda}$.

- ▶ The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}.$$

- ▶ Notice that $F(1) = c/(1 - e^{-\lambda}) = 1$.
- ▶ For probability distributions, we must always have $F(1) = 1$ since

$$F(1) = \sum_{k=0}^{\infty} P_k 1^k = \sum_{k=0}^{\infty} P_k = 1.$$

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Properties of generating functions

- ▶ Average degree:

$$\begin{aligned}\langle k \rangle &= \sum_{k=0}^{\infty} kP_k = \sum_{k=0}^{\infty} kP_k x^{k-1} \Big|_{x=1} \\ &= \frac{d}{dx} F(x) \Big|_{x=1} = F'(1)\end{aligned}$$

- ▶ In general, many calculations become simple, if a little abstract.
- ▶ For our exponential example:

$$F'(x) = \frac{(1 - e^{-\lambda})e^{-\lambda}}{(1 - xe^{-\lambda})^2}.$$

- ▶ So:

$$\langle k \rangle = F'(1) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})}.$$

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Properties of generating functions

Useful pieces for probability distributions:

- ▶ Normalization:

$$F(1) = 1$$

- ▶ First moment:

$$\langle k \rangle = F'(1)$$

- ▶ Higher moments:

$$\langle k^n \rangle = \left(x \frac{d}{dx} \right)^n F(x) \Big|_{x=1}$$

- ▶ k th element of sequence (general):

$$P_k = \frac{1}{k!} \frac{d^k}{dx^k} F(x) \Big|_{x=0}$$

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Edge-degree distribution

- ▶ Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

- ▶ Let's reëxpress our condition in terms of generating functions.
- ▶ We first need the g.f. for R_k .
- ▶ We'll now use this notation:

$F_P(x)$ is the g.f. for P_k .

$F_R(x)$ is the g.f. for R_k .

- ▶ Condition in terms of g.f. is:

$$\langle k \rangle_R = F'_R(1) > 1.$$

- ▶ Now find how F_R is related to F_P ...

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Edge-degree distribution

- ▶ We have

$$F_R(x) = \sum_{k=0}^{\infty} R_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)P_{k+1}}{\langle k \rangle} x^k.$$

Shift index to $j = k + 1$ and pull out $\frac{1}{\langle k \rangle}$:

$$\begin{aligned} F_R(x) &= \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} j P_j x^{j-1} = \frac{1}{\langle k \rangle} \sum_{j=1}^{\infty} P_j \frac{d}{dx} x^j \\ &= \frac{1}{\langle k \rangle} \frac{d}{dx} \sum_{j=1}^{\infty} P_j x^j = \frac{1}{\langle k \rangle} \frac{d}{dx} (F_P(x) - P_0) = \frac{1}{\langle k \rangle} F'_P(x). \end{aligned}$$

Finally, since $\langle k \rangle = F'_P(1)$,

$$F_R(x) = \frac{F'_P(x)}{F'_P(1)}$$

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Edge-degree distribution

- ▶ Recall giant component condition is $\langle k \rangle_R = F'_R(1) > 1$.
- ▶ Since we have $F_R(x) = F'_P(x)/F'_P(1)$,

$$F'_R(x) = \frac{F''_P(x)}{F'_P(1)}.$$

- ▶ Setting $x = 1$, our condition becomes

$$\frac{F''_P(1)}{F'_P(1)} > 1$$

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Size distributions

To figure out the **size of the largest component** (S_1), we need more resolution on component sizes.

Definitions:

- ▶ π_n = probability that a random node belongs to a finite component of size $n < \infty$.
- ▶ ρ_n = probability a random link leads to a finite subcomponent of size $n < \infty$.

Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

neighbors \Leftrightarrow components

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Size distributions

G.f.'s for component size distributions:



$$F_{\pi}(x) = \sum_{n=0}^{\infty} \pi_n x^n \quad \text{and} \quad F_{\rho}(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

The largest component:

- ▶ **Subtle key:** $F_{\pi}(1)$ is the probability that a node belongs to a **finite** component.
- ▶ Therefore: $S_1 = 1 - F_{\pi}(1)$.

Our mission, which we accept:

- ▶ Find the four generating functions

$$F_P, F_R, F_{\pi}, \text{ and } F_{\rho}.$$

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Useful results we'll need for g.f.'s

Sneaky Result 1:

- ▶ Consider two random variables U and V whose values may be $0, 1, 2, \dots$
- ▶ Write probability distributions as U_k and V_k and g.f.'s as F_U and F_V .
- ▶ **SR1**: If a third random variable is defined as

$$W = \sum_{i=1}^U V^{(i)} \text{ with each } V^{(i)} \stackrel{d}{=} V$$

then

$$F_W(x) = F_U(F_V(x))$$

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Proof of SR1:

Write probability that variable W has value k as W_k .

$$W_k = \sum_{j=0}^{\infty} U_j \times \Pr(\text{sum of } j \text{ draws of variable } V = k)$$

$$= \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} V_{i_2} \dots V_{i_j}$$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} U_j \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} V_{i_2} \dots V_{i_j} x^k$$

$$= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j}$$

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Proof of SR1:

With some concentration, observe:

$$\begin{aligned} F_W(x) &= \sum_{j=0}^{\infty} U_j \sum_{k=0}^{\infty} \sum_{\substack{\{i_1, i_2, \dots, i_j\} \\ i_1 + i_2 + \dots + i_j = k}} V_{i_1} x^{i_1} V_{i_2} x^{i_2} \dots V_{i_j} x^{i_j} \\ &\quad \underbrace{\hspace{15em}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j} \\ &\quad \left(\sum_{i'=0}^{\infty} V_{i'} x^{i'} \right)^j = (F_V(x))^j \\ &= \sum_{j=0}^{\infty} U_j (F_V(x))^j \\ &= F_U(F_V(x)) \checkmark \end{aligned}$$

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Useful results we'll need for g.f.'s

Sneaky Result 2:

- ▶ Start with a random variable U with distribution U_k ($k = 0, 1, 2, \dots$)
- ▶ **SR2:** If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

- ▶ **Reason:** $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.

▶

$$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

$$= x \sum_{j=0}^{\infty} U_j x^j = xF_U(x). \checkmark$$

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Generalization of SR2:

- ▶ (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

- ▶ (2) If $V = U - i$ then

$$F_V(x) = x^{-i} F_U(x)$$

$$= x^{-i} \sum_{k=0}^{\infty} U_k x^k$$

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Connecting generating functions

- ▶ **Goal:** figure out forms of the component generating functions, F_π and F_ρ .
- ▶ π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$



Therefore:

$$F_\pi(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_P(F_\rho(x))}_{\text{SR1}}$$

- ▶ Extra factor of x accounts for random node itself.

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Connecting generating functions

- ▶ ρ_n = probability that a random link leads to a finite subcomponent of size n .
- ▶ Invoke one step of recursion: ρ_n = probability that in following a random edge, the outgoing edges of the node reached lead to finite subcomponents of combined size $n - 1$,

$$= \sum_{k=0}^{\infty} R_k \times \Pr \left(\begin{array}{l} \text{sum of sizes of subcomponents} \\ \text{at end of } k \text{ random links} = n - 1 \end{array} \right)$$

▶

Therefore:

$$F_{\rho}(x) = \underbrace{x}_{\text{SR2}} \underbrace{F_R(F_{\rho}(x))}_{\text{SR1}}$$

- ▶ Again, extra factor of x accounts for random node itself.

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Connecting generating functions

- ▶ We now have two functional equations connecting our generating functions:

$$F_{\pi}(x) = xF_P(F_{\rho}(x)) \quad \text{and} \quad F_{\rho}(x) = xF_R(F_{\rho}(x))$$

- ▶ Taking stock: We know $F_P(x)$ and $F_R(x) = F'_P(x)/F'_P(1)$.
- ▶ We first untangle the **second equation** to find F_{ρ}
- ▶ We can do this because it **only involves** F_{ρ} and F_R .
- ▶ The first equation then immediately gives us F_{π} in terms of F_{ρ} and F_R .

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- ▶ Remembering vaguely what we are doing:

Finding F_π to obtain the **fractional size of the largest component** $S_1 = 1 - F_\pi(1)$.

- ▶ Set $x = 1$ in our two equations:

$$F_\pi(1) = F_P(F_\rho(1)) \quad \text{and} \quad F_\rho(1) = F_R(F_\rho(1))$$

- ▶ Solve second equation numerically for $F_\rho(1)$.
- ▶ Plug $F_\rho(1)$ into first equation to obtain $F_\pi(1)$.

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Example: Standard random graphs.

- ▶ We can show $F_P(x) = e^{-\langle k \rangle(1-x)}$

$$\therefore F_R(x) = F'_P(x)/F'_P(1) = e^{-\langle k \rangle(1-x)} / e^{-\langle k \rangle(1-x')} \Big|_{x'=1}$$

$$= e^{-\langle k \rangle(1-x)} = F_P(x) \quad \dots\text{aha!}$$

- ▶ RHS's of our two equations are the same.
- ▶ So $F_\pi(x) = F_\rho(x) = xF_R(F_\rho(x)) = xF_R(F_\pi(x))$
- ▶ Why our dirty (but wrong) trick worked earlier...

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Component sizes

- ▶ We are down to

$$F_{\pi}(x) = xF_R(F_{\pi}(x)) \text{ and } F_R(x) = e^{-\langle k \rangle(1-x)}.$$

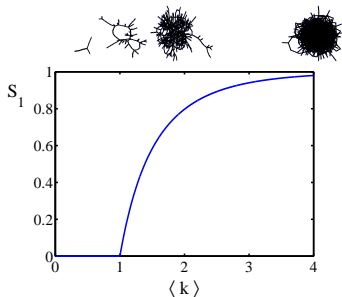


$$\therefore F_{\pi}(x) = xe^{-\langle k \rangle(1-F_{\pi}(x))}$$

- ▶ We're first after $S_1 = 1 - F_{\pi}(1)$ so set $x = 1$ and replace $F_{\pi}(1)$ by $1 - S_1$:

$$1 - S_1 = e^{-\langle k \rangle S_1}$$

$$\text{Or: } \langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$$



- ▶ Just as we found with our dirty trick ...
- ▶ Again, we (usually) have to resort to numerics ...

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- ▶ Next: find **average size** of finite components $\langle n \rangle$.
- ▶ Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.
- ▶ Try to avoid finding $F_\pi(x)$...
- ▶ Starting from $F_\pi(x) = xF_P(F_\rho(x))$, we differentiate:

$$F'_\pi(x) = F_P(F_\rho(x)) + xF'_\rho(x)F'_P(F_\rho(x))$$

- ▶ While $F_\rho(x) = xF_R(F_\rho(x))$ gives

$$F'_\rho(x) = F_R(F_\rho(x)) + xF'_\rho(x)F'_R(F_\rho(x))$$

- ▶ Now set $x = 1$ in both equations.
- ▶ We solve the second equation for $F'_\rho(1)$ (we must already have $F_\rho(1)$).
- ▶ Plug $F'_\rho(1)$ and $F_\rho(1)$ into first equation to find $F'_\pi(1)$.

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Example: Standard random graphs.

- ▶ Use fact that $F_P = F_R$ and $F_\pi = F_\rho$.
- ▶ Two differentiated equations reduce to only one:

$$F'_\pi(x) = F_P(F_\pi(x)) + xF'_\pi(x)F'_P(F_\pi(x))$$

$$\text{Rearrange: } F'_\pi(x) = \frac{F_P(F_\pi(x))}{1 - xF'_P(F_\pi(x))}$$

- ▶ Simplify denominator using $F'_P(x) = \langle k \rangle F_P(x)$
- ▶ Replace $F_P(F_\pi(x))$ using $F_\pi(x) = xF_P(F_\pi(x))$.
- ▶ Set $x = 1$ and replace $F_\pi(1)$ with $1 - S_1$.

$$\text{End result: } \langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$

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- ▶ Our result for standard random networks:

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$

- ▶ Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- ▶ Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- ▶ We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- ▶ This blows up as $\langle k \rangle \rightarrow 1$.
- ▶ **Reason:** we have a power law distribution of component sizes at $\langle k \rangle = 1$.
- ▶ Typical critical point behavior....

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- ▶ Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle(1 - S_1)}$$

- ▶ As $\langle k \rangle \rightarrow 0$, $S_1 = 0$, and $\langle n \rangle \rightarrow 1$.
- ▶ All nodes are isolated.
- ▶ As $\langle k \rangle \rightarrow \infty$, $S_1 \rightarrow 1$ and $\langle n \rangle \rightarrow 0$.
- ▶ No nodes are outside of the giant component.

Extra on largest component size:

- ▶ For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}$.
- ▶ For $\langle k \rangle < 1$, $S_1 \sim \log N$.

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