# Overview of Complex Networks

## Complex Networks, CSYS/MATH 303, Spring, 2010

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# Outline

# Class Admin

- Office hours:
  - Tuesday 1:00 pm–2:30 pm (Farrell Hall)
  - Appointments by email.
- Course outline
- Projects

# Exciting details regarding these slides:

- Three versions (all in pdf):
  1. Presentation,
  2. Flat Presentation,
  3. Handout (2x2).
- Presentation versions are navigable and hyperlinks are clickable.
- Web links look like this (⊞).
- References in slides link to full citation at end. [1]
- Citations contain links to papers in pdf (if available).
- Brought to you by a concoction of LaTeX, Beamer, and perl.

# Basic definitions

## Complex System—Some ingredients:

- Distributed system of many interrelated parts
- No centralized control
- Nonlinear relationships
- Existence of feedback loops
- Complex systems are open (out of equilibrium)
- Presence of Memory
- Modular (nested)/multiscale structure
- Opaque boundaries
- Emergence—'More is Different' [1]
- Many phenomena can be complex: social, technical, informational, geophysical, meteorological, fluidic, ...

# Basic definitions

Complex: (Latin = with + fold/weave (com + plex))

Adjective

- Made up of multiple parts; intricate or detailed.
- Not simple or straightforward.

net•work |ˈnetˌwərk|

noun

**1** an arrangement of intersecting horizontal and vertical lines.

• a complex system of roads, railroads, or other transportation routes : *a network of railroads.*

**2** a group or system of interconnected people or things : *a trade network.*

• a group of people who exchange information, contacts, and experience for professional or social purposes : *a support network.*

• a group of broadcasting stations that connect for the simultaneous broadcast of a program : *the introduction of a second TV network* | [as adj. ] *network television.*

• a number of interconnected computers, machines, or operations : *specialized computers that manage multiple outside connections to a network* | *a local cellular phone network.*

• a system of connected electrical conductors.

verb [ trans. ]

connect as or operate with a network : *the stock exchanges have proven to be resourceful in networking these deals.*

• link (machines, esp. computers) to operate interactively : [as adj. ] ( **networked**) *networked workstations.*

• [ intrans. ] [often as n. ] ( **networking**) interact with other people to exchange information and develop contacts, esp. to further one's career : *the skills of networking, bargaining, and negotiation.*

# Thesaurus deliciousness:

## network

noun

**1** *a network of arteries* WEB, lattice, net, matrix, mesh, crisscross, grid, reticulum, reticulation; Anatomy plexus.

**2** *a network of lanes* MAZE, labyrinth, warren, tangle.

**3** *a network of friends* SYSTEM, complex, nexus, web, webwork.

# Ancestry:

From Keith Briggs's excellent etymological investigation: (⊞)

- Opus reticulatum:
- A Latin origin?

[http://serialconsign.com/2007/11/we-put-net-network]

# Ancestry:

## First known use: Geneva Bible, 1560

'And thou shalt make unto it a grate like networke of brass (Exodus xxvii 4).'

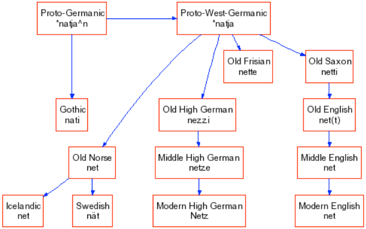
## From the OED via Briggs:

- 1658–: reticulate structures in animals
- 1839–: rivers and canals
- 1869–: railways
- 1883–: distribution network of electrical cables
- 1914–: wireless broadcasting networks

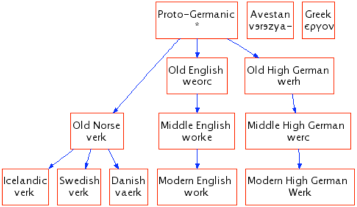
# Ancestry:

## Net and Work are venerable old words:

- ‘Net’ first used to mean spider web (King Ælfréd, 888).
- ‘Work’ appears to have long meant purposeful action.



The network of Germanic `net' words

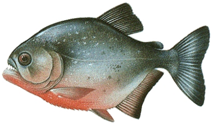


The network of `work' words

- ‘Network’ = something built based on the idea of natural, flexible lattice or web.
- c.f., ironwork, stonework, fretwork.

# Key Observation:

- Many complex systems
  can be viewed as complex networks
  of physical or abstract interactions.
- Opens door to mathematical and numerical analysis.
- Dominant approach of last decade of a theoretical-physics/stat-mechish flavor.
- Mindboggling amount of work published on complex networks since 1998...
- ... largely due to your typical theoretical physicist:



- *Piranha physicus*
- Hunt in packs.
- Feast on new and interesting ideas (see chaos, cellular automata, ...)

# Popularity (according to ISI)

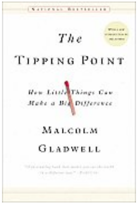
## “Collective dynamics of ‘small-world’ networks” [21]

- Watts and Strogatz
  Nature, 1998
- ≈ 4100 citations (as of January 18, 2010)
- Over 1100 citations in 2008 alone.

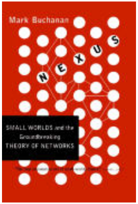
## “Emergence of scaling in random networks” [2]

- Barabási and Albert
  Science, 1999
- ≈ 4400 citations (as of January 18, 2010)
- Over 1100 citations in 2008 alone.

# Popularity according to books:

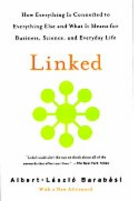



The Tipping Point: How Little Things can make a Big Difference—Malcolm Gladwell [9]

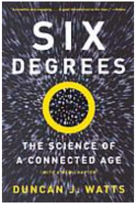



Nexus: Small Worlds and the Groundbreaking Science of Networks—Mark Buchanan

# Popularity according to books:




Linked: How Everything Is Connected to Everything Else and What It Means—Albert-Laszlo Barabási




Six Degrees: The Science of a Connected Age—Duncan Watts [19]

# Numerous others:

- Complex Social Networks—F. Vega-Redondo [18]
- Fractal River Basins: Chance and Self-Organization—I. Rodríguez-Iturbe and A. Rinaldo [15]
- Random Graph Dynamics—R. Durette
- Scale-Free Networks—Guido Caldarelli
- Evolution and Structure of the Internet: A Statistical Physics Approach—Romu Pastor-Satorras and Alessandro Vespignani
- Complex Graphs and Networks—Fan Chung
- Social Network Analysis—Stanley Wasserman and Kathleen Faust
- Handbook of Graphs and Networks—Eds: Stefan Bornholdt and H. G. Schuster [5]
- Evolution of Networks—S. N. Dorogovtsev and J. F. F. Mendes [8]

# More observations

- But surely networks aren't new...
- Graph theory is well established...
- Study of social networks started in the 1930's...
- So why all this 'new' research on networks?
- Answer: Oodles of Easily Accessible Data.
- We can now inform (alas) our theories with a much more measurable reality.*
- Real networks occupy a tiny, low entropy part of all network space and require specific attention.
- A worthy goal: establish mechanistic explanations.
- What kinds of dynamics lead to these real networks?

* *If this is upsetting, maybe string theory is for you...*

# More observations

- Web-scale data sets can be overly exciting.

## Witness:

- The End of Theory: The Data Deluge Makes the Scientific Theory Obsolete (Anderson, Wired) (⊞)
- “The Unreasonable Effectiveness of Data,” Halevy et al. [10]
- c.f. Wigner’s “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” [22]

## But:

- For scientists, description is only part of the battle.
- We still need to understand.

# Super basic definitions

Nodes = A collection of entities which have properties that are somehow related to each other

- e.g., people, forks in rivers, proteins, webpages, organisms,...

# Basic definitions

**Links = Connections between nodes**

- links
  - may be real and fixed (rivers),
  - real and dynamic (airline routes),
  - abstract with physical impact (hyperlinks),
  - or purely abstract (semantic connections between concepts).
- Links may be directed or undirected.
- Links may be binary or weighted.

# Basic definitions





## Node degree = Number of links per node

- Notation: Node $i$'s degree = $k_i$.
- $k_i = 0,1,2,\ldots$
- Notation: the average degree of a network = $\langle k \rangle$ (and sometimes as $z$)
- For undirected networks, connection between number of edges $m$ and average degree:

$$\langle k \rangle = \frac{2m}{N}$$

- For directed networks,

$$\langle k_{\mathrm{out}} \rangle = \langle k_{\mathrm{in}} \rangle = \frac{m}{N}$$

- Defn: $\mathcal{N}_i$ = the set of $i$'s $k_i$ neighbors

# Basic definitions

## Adjacency matrix:

- We represent a graph or network by a matrix $A$ with link weight $a_{ij}$ for nodes $i$ and $j$ in entry $(i, j)$.
- e.g.,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

- (n.b., for numerical work, we always use sparse matrices.)

# Examples

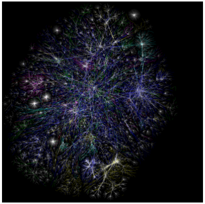
## What passes for a complex network?

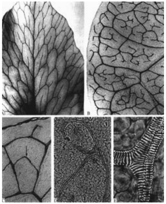
- Complex networks are large (in node number)
- Complex networks are sparse (low edge to node ratio)
- Complex networks are usually dynamic and evolving
- Complex networks can be social, economic, natural, informational, abstract, ...

# Examples

## Physical networks

- River networks
- Neural networks
- Trees and leaves
- Blood networks
- The Internet
- Road networks
- Power grids





- Distribution (branching) versus redistribution (cyclical)

# Examples

## Interaction networks

- The Blogosphere
- Biochemical networks
- Gene-protein networks
- Food webs: who eats whom
- The World Wide Web (?)
- Airline networks
- Call networks (AT&T)
- The Media

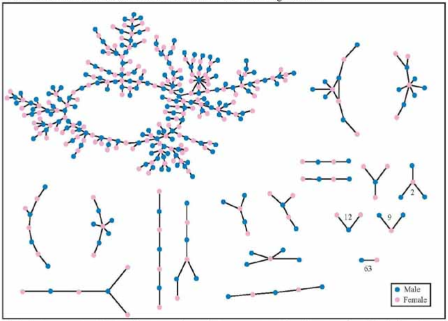
datamining.typepad.com (⊞)

# Examples

## Interaction networks: social networks

- Snogging
- Friendships
- Acquaintances
- Boards and directors
- Organizations
- myspace.com (⊞), facebook.com (⊞)

The Structure of Romantic and Sexual Relations at "Jefferson High School"



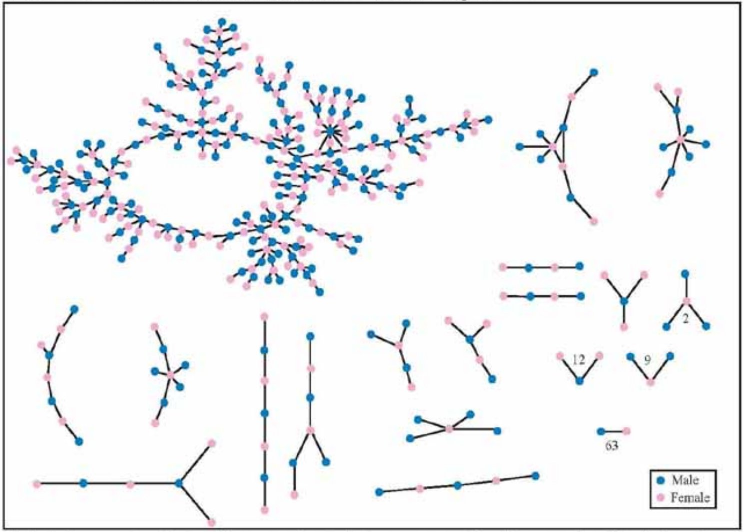

Each circle represents a student and lines connecting students represent romantic relations occuring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).

(Bearman *et al.*, 2004)

- 'Remotely sensed' by: email activity, instant messaging, phone logs (*cough*).

# Examples

The Structure of Romantic and Sexual Relations at "Jefferson High School"




Each circle represents a student and lines connecting students represent romantic relations occuring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).

# Examples

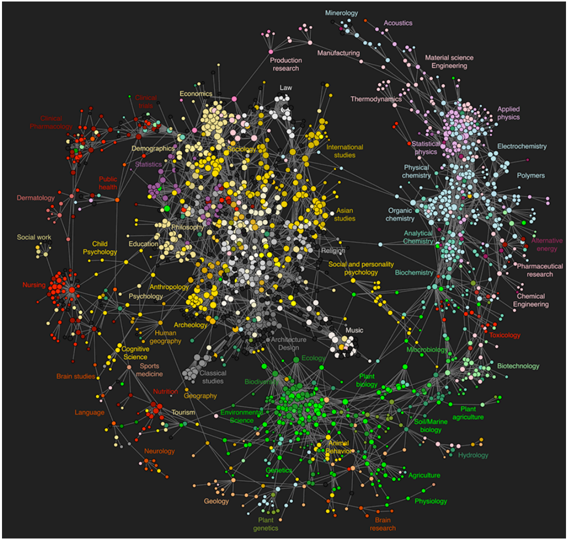
## Relational networks

- Consumer purchases
  (Wal-Mart: $\approx$ 1 petabyte $= 10^{15}$ bytes)
- Thesauri: Networks of words generated by meanings
- Knowledge/Databases/Ideas
- Metadata—Tagging: del.icio.us (⊞), flickr (⊞)

**common tags** cloud | list

community daily dictionary education **encyclopedia**
english free imported info information internet knowledge
learning news **reference** research resource
resources search tools useful web web2.0 **wiki**
**wikipedia**

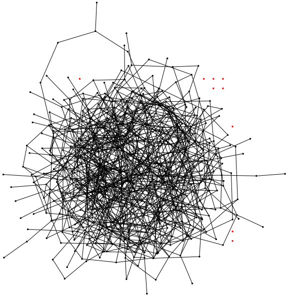
# Clickworthy Science:




Bollen et al. [4]

# A notable feature of large-scale networks:

- Graphical renderings are often just a big mess.



$\Leftarrow$ Typical hairball

- number of nodes $N = 500$
- number of edges $m = 1000$
- average degree $\langle k \rangle = 4$

- And even when renderings somehow look good:
  "That is a very graphic analogy which aids understanding wonderfully while being, strictly speaking, wrong in every possible way"
  said Ponder [Stibbons] —*Making Money*, T. Pratchett.
- We need to extract digestible, meaningful aspects.

# Properties

Some key aspects of real complex networks:

- degree distribution
- assortativity
- homophily
- clustering
- motifs
- modularity
- concurrency
- hierarchical scaling
- network distances
- centrality
- efficiency
- robustness

- + Coevolution of network structure and processes on networks.

# Properties

## 1. degree distribution $P_k$

- $P_k$ is the probability that a randomly selected node has degree $k$
- $k$ = node degree = number of connections
- ex 1: Erdős-Rényi random networks:

$$P_k = e^{-\langle k \rangle} \langle k \rangle^k / k!$$

- Distribution is Poisson

# Properties

1. degree distribution $P_k$
   - ex 2: "Scale-free" networks: $P_k \propto k^{-\gamma} \Rightarrow$ 'hubs'
   - link cost controls skew
   - hubs may facilitate or impede contagion

# Properties

## Note:

- Erdős-Rényi random networks are a *mathematical construct*.
- 'Scale-free' networks are growing networks that form according to a plausible mechanism.
- Randomness is out there, just not to the degree of a completely random network.

# Properties

## 2. assortativity/3. homophily:

- Social networks: Homophily (⊞) = birds of a feather
- e.g., degree is standard property for sorting: measure degree-degree correlations.
- Assortative network:[13] similar degree nodes connecting to each other.
  *Often social: company directors, coauthors, actors.*
- Disassortative network: high degree nodes connecting to low degree nodes.
  *Often techological or biological: Internet, WWW, protein interactions, neural networks, food webs.*

# Clustering

## 4. clustering:

- Your friends tend to know each other.
- Two measures:
  1. Watts & Strogatz [21]

$$C_1 = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i$$

  2. Newman [14]

$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$

# Properties

## First clustering measure:

- $C_1$ is the average fraction of pairs of neighbors who are connected.
- Fraction of pairs of neighbors who are connected is

$$\frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2}$$

  where $k_i$ is node $i$'s degree, and $\mathcal{N}_i$ is the set of $i$'s neighbors.
- Averaging over all nodes, we have

$$C_1 = \frac{1}{n}\sum_{i=1}^{n} \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i - 1)/2} \right\rangle_i$$

# Properties

- For sparse networks, $C_1$ tends to discount highly connected nodes.
- $C_2$ is a useful and often preferred variant
- In general, $C_1 \neq C_2$.
- $C_1$ is a global average of a local ratio.
- $C_2$ is a ratio of two global quantities.

# Properties

## Triples and triangles

- Nodes $i_1$, $i_2$, and $i_3$ form a triple around $i_1$ if $i_1$ is connected to $i_2$ and $i_3$.
- Nodes $i_1$, $i_2$, and $i_3$ form a triangle if each pair of nodes is connected
- The definition

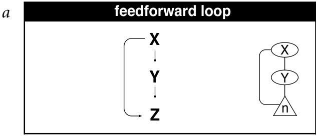
$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$

  measures the fraction of closed triples
- Social Network Analysis (SNA): fraction of transitive triples.
- The '3' appears because for each triangle, we have 3 closed triples.

# Properties

## 5. motifs:

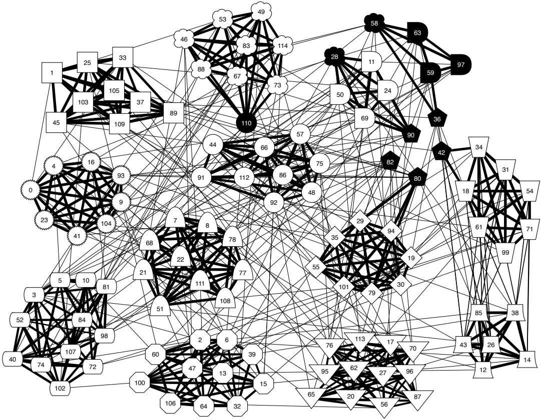
- small, recurring functional subnetworks
- e.g., Feed Forward Loop:




Shen-Orr, Uri Alon, *et al.* [16]

# Properties

## 6. modularity and structure/community detection:




Clauset *et al.*, 2006 [7]: NCAA football

# Properties

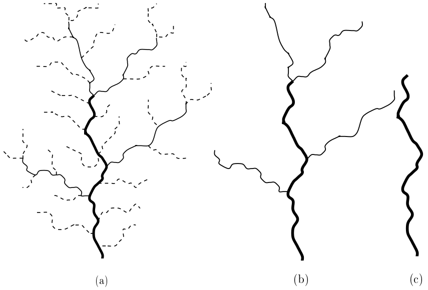
7. concurrency:

- transmission of a contagious element only occurs during contact
- rather obvious but easily missed in a simple model
- dynamic property—static networks are not enough
- knowledge of previous contacts crucial
- beware cumulated network data
- Kretzschmar and Morris, 1996 [12]

# Properties

## 8. Horton-Strahler ratios:

- Metrics for branching networks:
  - Method for ordering streams hierarchically
  - Number: $R_n = N_\omega / N_{\omega+1}$
  - Segment length: $R_l = \langle l_{\omega+1} \rangle / \langle l_\omega \rangle$
  - Area/Volume: $R_a = \langle a_{\omega+1} \rangle / \langle a_\omega \rangle$



# Properties

9. network distances:

(a) shortest path length $d_{ij}$:

- Fewest number of steps between nodes $i$ and $j$.
- (Also called the chemical distance between $i$ and $j$.)

(b) average path length $\langle d_{ij} \rangle$:

- Average shortest path length in whole network.
- Good algorithms exist for calculation.
- Weighted links can be accommodated.

# Properties

9. network distances:

- network diameter $d_{\max}$:
  Maximum shortest path length between any two nodes.
- closeness $d_{\mathrm{cl}} = [\sum_{ij} d_{ij}{}^{-1} / \binom{n}{2}]^{-1}$:
  Average 'distance' between any two nodes.
- Closeness handles disconnected networks ($d_{ij} = \infty$)
- $d_{\mathrm{cl}} = \infty$ only when all nodes are isolated.
- Closeness perhaps compresses too much into one number

# Properties

## 10. centrality:

- Many such measures of a node's 'importance.'
- ex 1: Degree centrality: $k_i$.
- ex 2: Node $i$'s betweenness
  = fraction of shortest paths that pass through $i$.
- ex 3: Edge $\ell$'s betweenness
  = fraction of shortest paths that travel along $\ell$.
- ex 4: Recursive centrality: Hubs and Authorities (Jon Kleinberg [11])

# Models

Some important models:

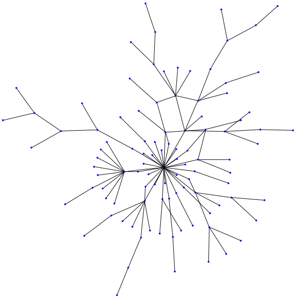
1. generalized random networks (touched on in 300)
2. scale-free networks (⊞) (covered in 300)
3. small-world networks (⊞) (covered in 300)
4. statistical generative models ($p^*$)
5. generalized affiliation networks (partly covered in 300)

# Models

1. generalized random networks:
   - Arbitrary degree distribution $P_k$.
   - Wire nodes together randomly.
   - Create ensemble to test deviations from randomness.
   - Interesting, applicable, rich mathematically.
   - We will have fun with these guys...

# Models

## 2. 'scale-free networks':



$\gamma$ = 2.5
$\langle k \rangle$ = 1.8
$N = 150$

- Introduced by Barabasi and Albert [2]
- Generative model
- Preferential attachment model with growth:
- $P$[attachment to node i] $\propto \mathrm{k_i^\alpha}$.
- Produces $P_k \sim k^{-\gamma}$ when $\alpha = 1$.
- Trickiness: other models generate skewed degree distributions.

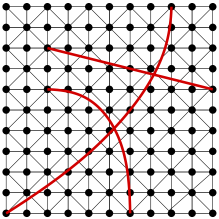
# Models

## 3. small-world networks

- Introduced by Watts and Strogatz [21]

Two scales:

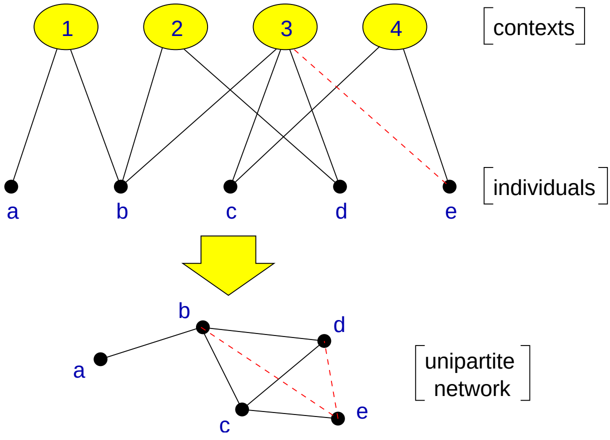
- local regularity (an individual's friends know each other)
- global randomness (shortcuts).



- Shortcuts allow disease to jump
- Number of infectives increases exponentially in time
- Facilitates synchronization

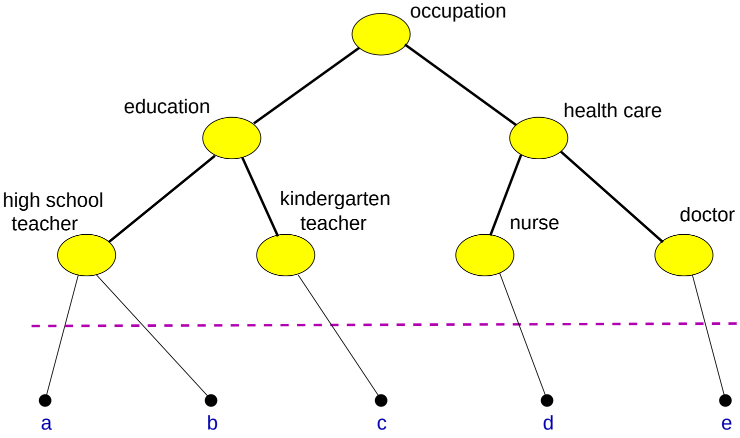
# Models

## 5. generalized affiliation networks




Bipartite affiliation networks: boards and directors, movies and actors.

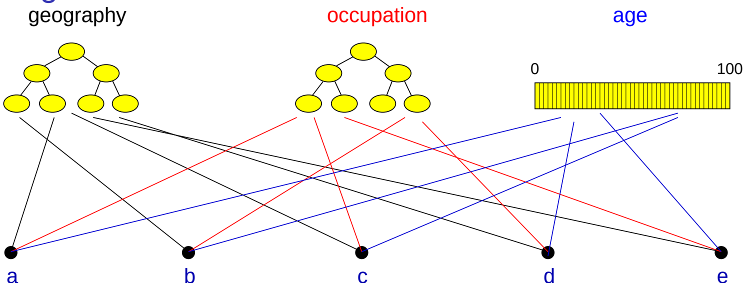
## 5. generalized affiliation networks



# Models



## 5. generalized affiliation networks




- Blau & Schwartz [3], Simmel [17], Breiger [6], Watts *et al.* [20]

# Nutshell:

## Overview Key Points:

- The field of complex networks came into existence in the late 1990s.
- Explosion of papers and interest since 1998/99.
- Hardened up much thinking about complex systems.
- Specific focus on networks that are large-scale, sparse, natural or man-made, evolving and dynamic, and (crucially) measurable.
- Three main (blurred) categories:
  1. Physical (e.g., river networks),
  2. Interactional (e.g., social networks),
  3. Abstract (e.g., thesauri).

# Nutshell:

## Overview Key Points (cont.):

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
  1. Description: Characterizing very large networks
  2. Explanation: Micro story $\Rightarrow$ Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- Still much work to be done, especially with respect to dynamics... exciting!

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