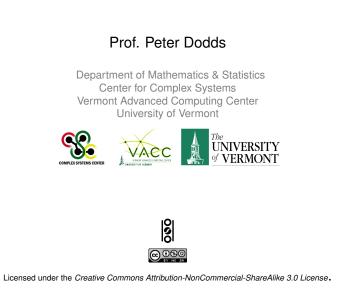
Random walks and diffusion on networks Complex Networks, CSYS/MATH 303, Spring, 2010



Random walks on networks—basics:

- Imagine a single random walker moving around on a network.
- At t = 0, start walker at node *j* and take time to be discrete.
- Q: What's the long term probability distribution for where the walker will be?
- Define $p_i(t)$ as the probability that at time step t, our walker is at node *i*.
- We want to characterize the evolution of $\vec{p}(t)$.
- First task: connect $\vec{p}(t+1)$ to $\vec{p}(t)$.
- ► Let's call our walker Barry.
- Unfortunately for Barry, he lives on a high dimensional graph and is far from home.
- Worse still: Barry is hopelessly drunk.

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Consider simple undirected networks with an edges either present of absent.

Represent network by a symmetric adjacency matrix A where

> $a_{ii} = 1$ if *i* and *j* are connected, $a_{ii} = 0$ otherwise.

- Barry is at node *i* at time *t* with probability $p_i(t)$.
- In the next time step he randomly lurches toward one of i's neighbors.
- Equation-wise:

Where is Barry?

$$p_j(t+1) = \sum_{i=1}^n \frac{1}{k_i} a_{ji} p_i(t).$$

where k_i is *i*'s degree. Note: $k_i = \sum_{i=1}^n a_{ii}$.

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Where is Barry?

• Linear algebra-based excitement: $p(t+1) = \sum_{i=1}^{n} 1 = p(t)$ is more use

 $p_j(t+1) = \sum_{i=1}^n \frac{1}{k_i} a_{ji} p_i(t)$ is more usefully viewed as

 $\vec{p}(t+1) = A^{\mathrm{T}} K^{-1} \vec{p}(t)$

where $[K_{ij}] = [\delta_{ij}k_i]$ has node degrees on the main diagonal and zeros everywhere else.

- So... we need to find the dominant eigenvalue of A^TK⁻¹.
- Expect this eigenvalue will be 1 (doesn't make sense for total probability to change).
- The corresponding eigenvector will be the limiting probability distribution (or invariant measure).
- Extra concerns: multiplicity of eigenvalue = 1, and network connectedness.

Other pieces:

- ► Good news: A^TK⁻¹ is similar to a real symmetric matrix.
- Consider the transformation $M = K^{-1/2}$:

$$K^{-1/2} \mathbf{A}^{\mathrm{T}} \mathbf{K}^{-1} \mathbf{K}^{1/2} = K^{-1/2} \mathbf{A}^{\mathrm{T}} \mathbf{K}^{-1/2}$$

• Since $A^{\mathrm{T}} = A$, we have

$$(K^{-1/2}AK^{-1/2})^{\mathrm{T}} = K^{-1/2}AK^{-1/2}$$

- Upshot: A^TK⁻¹ has real eigenvalues and a complete set of orthogonal eigenvectors.
- Can also show that maximum eigenvalue magnitude is indeed 1.
- Other goodies: next time round.

Where is Barry?

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By inspection, we see that

$$\vec{p}(\infty) = rac{1}{\sum_{i=1}^{n} k_i} \bar{k}$$

satisfies $\vec{p}(\infty) = A^{T} K^{-1} \vec{p}(\infty)$ with eigenvalue 1.

- We will find Barry at node *i* with probability proportional to its degree k_i.
- Nice implication: probability of finding Barry travelling along any edge is uniform.
- Diffusion in real space smooths things out.
- On networks, uniformity occurs on edges.
- So in fact, diffusion in real space is about the edges too but we just don't see that.

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