Branching Networks II

Complex Networks, CSYS/MATH 303, Spring, 2010

Prof. Peter Dodds

Department of Mathematics & Statistics
Center for Complex Systems
Vermont Advanced Computing Center
University of Vermont

Outline

Can Horton and Tokunaga be happy?

Horton and Tokunaga seem different:

- In terms of network achitecture, Horton's laws appear to contain less detailed information than Tokunaga's law.
- Oddly, Horton's laws have four parameters and Tokunaga has two parameters.
- R_n, R_a, R_ℓ, and R_s versus T_1 and R_T. One simple redundancy: $R_\ell = R_s$.
  Insert question 2, assignment 2 (⊞)
- To make a connection, clearest approach is to start with Tokunaga's law...
- Known result: Tokunaga → Horton [18, 19, 20, 9, 2]

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Let us make them happy

We need one more ingredient:

Space-fillingness

- A network is space-filling if the average distance between adjacent streams is roughly constant.
- Reasonable for river and cardiovascular networks
- For river networks:
  Drainage density ρ_{dd} = inverse of typical distance between channels in a landscape.
- In terms of basin characteristics:

$$\rho_{\mathrm{dd}} \simeq \frac{\sum \text{stream segment lengths}}{\text{basin area}}$$

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More with the happy-making thing

Start with Tokunaga's law: $T_k = T_1 R_T^{k-1}$

- Start looking for Horton's stream number law: $n_\omega / n_{\omega+1} = R_n$.
- Estimate n_ω, the number of streams of order ω in terms of other $n_{\omega'}$, $\omega' > \omega$.
- Observe that each stream of order ω terminates by either:

1. Running into another stream of order ω and generating a stream of order $\omega + 1$...
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     - $2n_{\omega+1}$ streams of order ω do this
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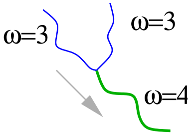

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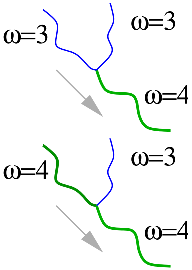

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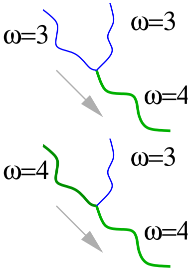

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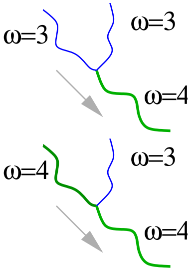

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Putting things together:

- $$n_\omega = \underbrace{2 n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega} n_{\omega'}}_{\text{absorption}}$$
- Use Tokunaga's law and manipulate expression to create R_n's.
- Insert question 3, assignment 2 (⊞)
- Solution:

$$R_n = \frac{(2 + R_T + T_1) \pm \sqrt{(2 + R_T + T_1)^2 - 8 R_T}}{2}$$

(The larger value is the one we want.)

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Finding other Horton ratios

Connect Tokunaga to R_s

- Now use uniform drainage density ρ_{dd}.
- Assume side streams are roughly separated by distance $1/\rho_{\mathrm{dd}}$.
- For an order ω stream segment, expected length is

$$\bar{s}_\omega \simeq \rho_{\mathrm{dd}}^{-1}\left(1 + \sum_{k=1}^{\omega-1} T_k\right)$$

- Substitute in Tokunaga's law $T_k = T_1 R_T^{k-1}$:

$$\bar{s}_\omega \simeq \rho_{\mathrm{dd}}^{-1}\left(1 + T_1 \sum_{k=1}^{\omega-1} R_T^{k-1}\right) \propto R_T^{\omega}$$

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Altogether then:

- $$\Rightarrow \bar{s}_\omega / \bar{s}_{\omega-1} = R_T \Rightarrow R_s = R_T$$

- Recall $R_\ell = R_s$ so

$$\boxed{R_\ell = R_T}$$

- And from before:

$$\boxed{R_n = \frac{(2 + R_T + T_1) + \sqrt{(2 + R_T + T_1)^2 - 8R_T}}{2}}$$

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Some observations:

- R_n and R_ℓ depend on T_1 and R_T.
- Seems that R_a must as well...
- Suggests Horton's laws must contain some redundancy
- We'll in fact see that $R_a = R_n$.
- Also: Both Tokunaga's law and Horton's laws can be generalized to relationships between non-trivial statistical distributions. [3, 4]

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The other way round

- Note: We can invert the expresssions for R_n and R_ℓ to find Tokunaga's parameters in terms of Horton's parameters.
- $$R_T = R_\ell,$$
- $$T_1 = R_n - R_\ell - 2 + 2R_\ell/R_n.$$
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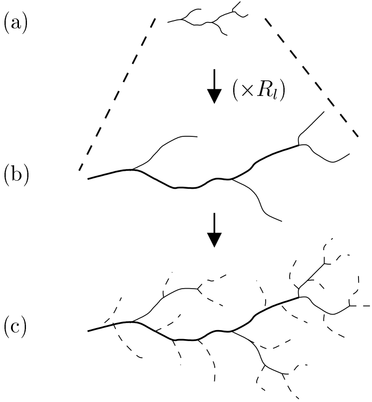

- Assume Horton's laws hold for number and length
- Start with an order ω stream
- Scale up by a factor of R_ℓ, orders increment
- Maintain drainage density by adding new order 1 streams

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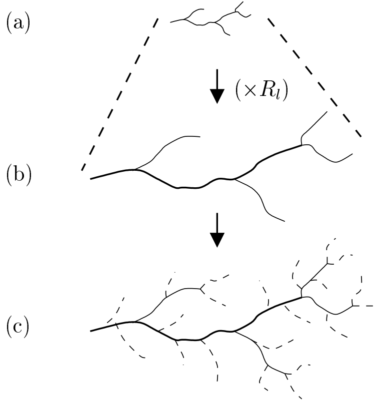

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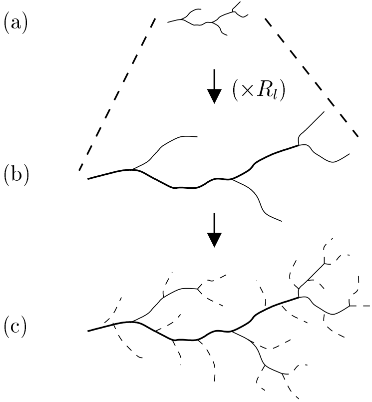

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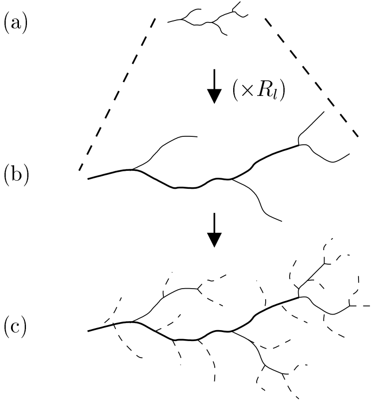

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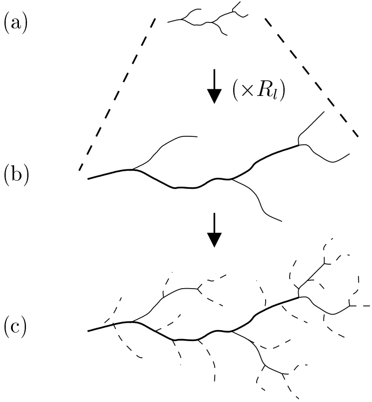

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...and in detail:

- Must retain same drainage density.
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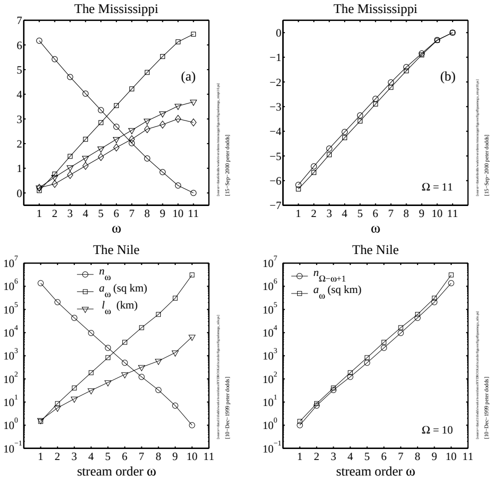

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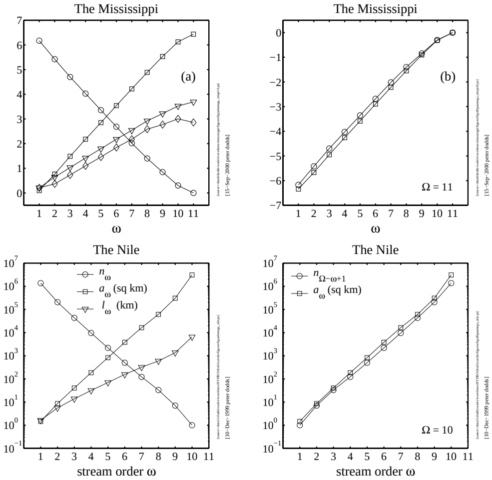

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ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
$[2,3]$	5.27	5.26	2.48	2.30	1.00
$[2,5]$	4.86	4.96	2.42	2.31	1.02
$[2,7]$	4.77	4.88	2.40	2.31	1.02
$[3,4]$	4.72	4.91	2.41	2.34	1.04
$[3,6]$	4.70	4.83	2.40	2.35	1.03
$[3,8]$	4.60	4.79	2.38	2.34	1.04
$[4,6]$	4.69	4.81	2.40	2.36	1.02
$[4,8]$	4.57	4.77	2.38	2.34	1.05
$[5,7]$	4.68	4.83	2.36	2.29	1.03
$[6,7]$	4.63	4.76	2.30	2.16	1.03
$[7,8]$	4.16	4.67	2.41	2.56	1.12
mean μ	4.69	4.85	2.40	2.33	1.04
std dev σ	0.21	0.13	0.04	0.07	0.03
σ/μ	0.045	0.027	0.015	0.031	0.024

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ω range	R_n	R_a	R_ℓ	R_s	R_a/R_n
[2, 3]	4.78	4.71	2.47	2.08	0.99
[2, 5]	4.55	4.58	2.32	2.12	1.01
[2, 7]	4.42	4.53	2.24	2.10	1.02
[3, 5]	4.45	4.52	2.26	2.14	1.01
[3, 7]	4.35	4.49	2.20	2.10	1.03
[4, 6]	4.38	4.54	2.22	2.18	1.03
[5, 6]	4.38	4.62	2.22	2.21	1.06
[6, 7]	4.08	4.27	2.05	1.83	1.05
mean μ	4.42	4.53	2.25	2.10	1.02
std dev σ	0.17	0.10	0.10	0.09	0.02
σ/μ	0.038	0.023	0.045	0.042	0.019

Reducing Horton's laws:

Rough first effort to show $R_n \equiv R_a$:

- $a_\Omega \propto$ sum of all stream lengths in a order Ω basin (assuming uniform drainage density)
- So:

$$a_\Omega \simeq \sum_{\omega=1}^{\Omega} n_\omega \bar{s}_\omega / \rho_{\mathrm{dd}}$$

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Intriguing division of area:

- Observe: Combined area of basins of order ω independent of ω.
- Not obvious: basins of low orders not necessarily contained in basis on higher orders.
- Story:

$$R_n \equiv R_a \Rightarrow \boxed{n_\omega \bar{a}_\omega = \text{const}}$$

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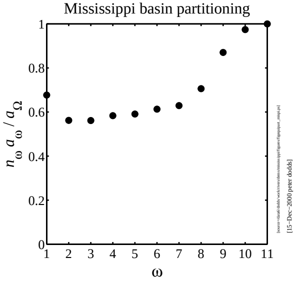

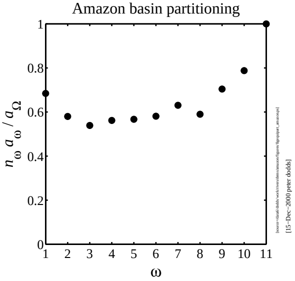

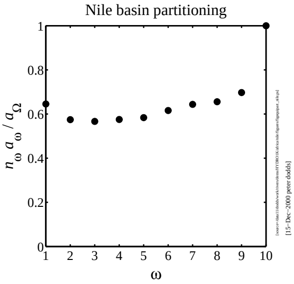

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The story so far:

- Natural branching networks are hierarchical, self-similar structures
- Hierarchy is mixed
- Tokunaga's law describes detailed architecture: $T_k = T_1 R_T^{k-1}$.
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A little further...

- Ignore stream ordering for the moment
- Pick a random location on a branching network p.
- Each point p is associated with a basin and a longest stream length
- Q: What is probability that the p's drainage basin has area a? $P(a) \propto a^{-\tau}$ for large a
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- Q: What is probability that the *p*'s drainage basin has area *a*? $P(a) \propto a^{-\tau}$ for large *a*
- Q: What is probability that the longest stream from *p* has length ℓ? $P(\ell) \propto \ell^{-\gamma}$ for large ℓ
- Roughly observed: $1.3 \lesssim \tau \lesssim 1.5$ and $1.7 \lesssim \gamma \lesssim 2.0$

Scaling laws

A little further...

- Ignore stream ordering for the moment
- Pick a random location on a branching network p.
- Each point p is associated with a basin and a longest stream length
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Scaling laws

Probability distributions with power-law decays

- We see them everywhere:
  - Earthquake magnitudes (Gutenberg-Richter law)
  - City sizes (Zipf's law)
  - Word frequency (Zipf's law) [21]
  - Wealth (maybe not—at least heavy tailed)
  - Statistical mechanics (phase transitions) [5]
- A big part of the story of complex systems
- Arise from mechanisms: growth, randomness, optimization, ...
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Connecting exponents

- We have the detailed picture of branching networks (Tokunaga and Horton)
- Plan: Derive $P(a) \propto a^{-\tau}$ and $P(\ell) \propto \ell^{-\gamma}$ starting with Tokunaga/Horton story [17, 1, 2]
- Let's work on $P(\ell)$...
- Our first fudge: assume Horton's laws hold throughout a basin of order Ω.
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Finding γ:

- Often useful to work with cumulative distributions, especially when dealing with power-law distributions.
- The complementary cumulative distribution turns out to be most useful:

$$P_>(\ell_*) = P(\ell > \ell_*) = \int_{\ell=\ell_*}^{\ell_{\max}} P(\ell)\mathrm{d}\ell$$

- $$P_>(\ell_*) = 1 - P(\ell < \ell_*)$$
- Also known as the exceedance probability.

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Finding γ:

- The connection between $P(x)$ and $P_>(x)$ when $P(x)$ has a power law tail is simple:
- Given $P(\ell) \sim \ell^{-\gamma}$ large ℓ then for large enough ℓ_*

$$P_>(\ell_*) = \int_{\ell=\ell_*}^{\ell_{\max}} P(\ell)\,\mathrm{d}\ell$$

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Scaling laws

Finding γ:

- Aim: determine probability of randomly choosing a point on a network with main stream length $> \ell_*$
- Assume some spatial sampling resolution Δ
- Landscape is broken up into grid of $\Delta \times \Delta$ sites
- Approximate $P_>(\ell_*)$ as

$$P_>(\ell_*) = \frac{N_>(\ell_*;\Delta)}{N_>(0;\Delta)}.$$

  where $N_>(\ell_*;\Delta)$ is the number of sites with main stream length $> \ell_*$.
- Use Horton's law of stream segments: $s_\omega / s_{\omega-1} = R_s$...

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$$P_>(\ell_\omega) = \frac{N_>(\ell_\omega; \Delta)}{N_>(0; \Delta)} \simeq \frac{\sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} / \Delta}{\sum_{\omega'=1}^{\Omega} n_{\omega'} s_{\omega'} / \Delta}$$
- Δ's cancel
- Denominator is $a_\Omega \rho_{\mathrm{dd}}$, a constant.
- So... using Horton's laws...

$$P_>(\ell_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} n_{\omega'} s_{\omega'} \simeq \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'})(\bar{s}_1 \cdot R_s^{\omega'-1})$$

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$$P_>(\ell_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} (1 \cdot R_n^{\Omega-\omega'})(\bar{s}_1 \cdot R_s^{\omega'-1})$$

- Cleaning up irrelevant constants:

$$P_>(\ell_\omega) \propto \sum_{\omega'=\omega+1}^{\Omega} \left(\frac{R_s}{R_n}\right)^{\omega'}$$

- Change summation order by substituting $\omega'' = \Omega - \omega'$.
- Sum is now from $\omega'' = 0$ to $\omega'' = \Omega - \omega - 1$ (equivalent to $\omega' = \Omega$ down to $\omega' = \omega + 1$)

Scaling laws

Finding γ:

- $$P_{>}(\ell_\omega) \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_s}{R_n}\right)^{\Omega-\omega''} \propto \sum_{\omega''=0}^{\Omega-\omega-1} \left(\frac{R_n}{R_s}\right)^{\omega''}$$

- Since $R_n > R_s$ and $1 \ll \omega \ll \Omega$,

$$P_{>}(\ell_\omega) \propto \left(\frac{R_n}{R_s}\right)^{\Omega-\omega} \propto \left(\frac{R_n}{R_s}\right)^{-\omega}$$

again using $\sum_{i=0}^{n-1} a^i = (a^n - 1)/(a-1)$

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- Need to express right hand side in terms of ℓ_ω.
- Recall that $\ell_\omega \simeq \bar{\ell}_1 R_\ell^{\omega-1}$.
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Scaling laws

Finding γ:

- And so we have:

$$\gamma = \ln R_n / \ln R_s$$

- Proceeding in a similar fashion, we can show

$$\tau = 2 - \ln R_s / \ln R_n = 2 - 1/\gamma$$

Insert question 5, assignment 2 (⊞)

- Such connections between exponents are called scaling relations
- Let's connect to one last relationship: Hack's law

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Hack's law: [6]

- $$\ell \propto a^h$$
- Typically observed that $0.5 \lesssim h \lesssim 0.7$.
- Use Horton laws to connect h to Horton ratios:

$$\ell_\omega \propto R_s^\omega \text{ and } a_\omega \propto R_n^\omega$$

- Observe:

$$\ell_\omega \propto e^{\omega \ln R_s} \propto \left(e^{\omega \ln R_n}\right)^{\ln R_s / \ln R_n}$$

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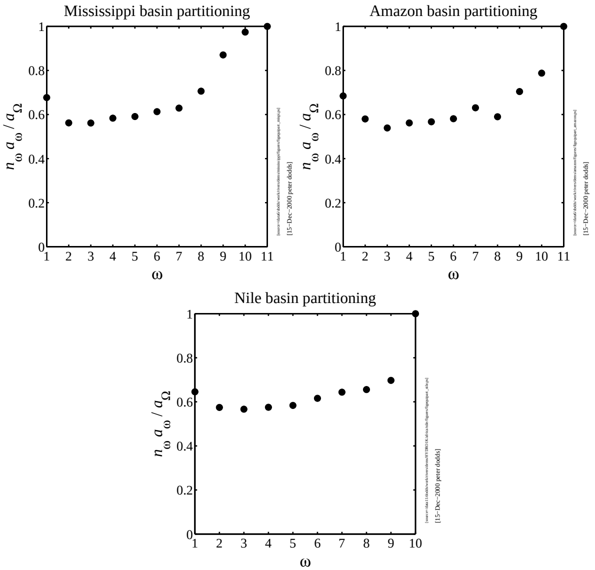
Connecting exponents

Only 3 parameters are independent:
e.g., take d, R_n, and R_s

relation:	scaling relation/parameter: [2]
$\ell \sim L^d$	d
$T_k = T_1 (R_T)^{k-1}$	$T_1 = R_n - R_s - 2 + 2R_s/R_n$
	$R_T = R_s$
$n_\omega / n_{\omega+1} = R_n$	R_n
$\bar{a}_{\omega+1} / \bar{a}_\omega = R_a$	$R_a = R_n$
$\bar{\ell}_{\omega+1} / \bar{\ell}_\omega = R_\ell$	$R_\ell = R_s$
$\ell \sim a^h$	$h = \log R_s / \log R_n$
$a \sim L^D$	$D = d/h$
$L_\perp \sim L^H$	$H = d/h - 1$
$P(a) \sim a^{-\tau}$	$\tau = 2 - h$
$P(\ell) \sim \ell^{-\gamma}$	$\gamma = 1/h$
$\Lambda \sim a^\beta$	$\beta = 1 + h$
$\lambda \sim L^\varphi$	$\varphi = d$

Equipartitioning reexamined:

Recall this story:



Equipartitioning

- What about

$$P(a) \sim a^{-\tau} \qquad ?$$

- Since $\tau > 1$, suggests no equipartitioning:

$$aP(a) \sim a^{-\tau+1} \neq \text{const}$$

- $P(a)$ overcounts basins within basins...
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Fluctuations

Moving beyond the mean:

- Both Horton's laws and Tokunaga's law relate average properties, e.g.,

$$\bar{s}_\omega / \bar{s}_{\omega-1} = R_s$$

- Natural generalization to consideration relationships between probability distributions
- Yields rich and full description of branching network structure
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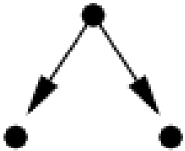
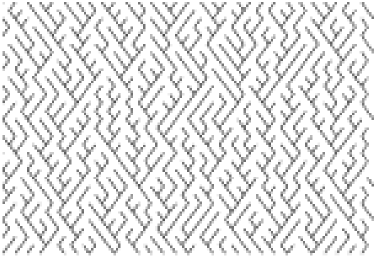
A toy model—Scheidegger's model

Directed random networks [11, 12]





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-

$$P(\searrow) = P(\swarrow) = 1/2$$

- Flow is directed downwards
- Useful and interesting test case—more later...

Horton ⇔ Tokunaga
Reducing Horton
Scaling relations
Fluctuations
Models
Nutshell
References

Generalizing Horton's laws

- $\bar{\ell}_\omega \propto (R_\ell)^\omega \Rightarrow N(\ell|\omega) = (R_n R_\ell)^{-\omega} F_\ell(\ell/R_\ell^\omega)$
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- Scaling collapse works well for intermediate orders
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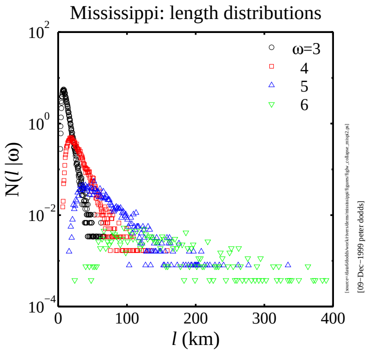

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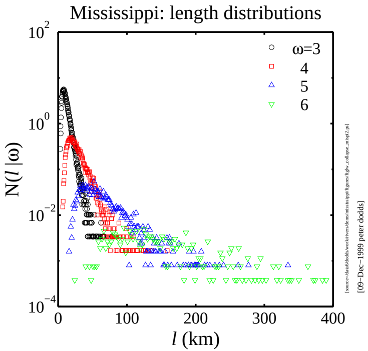

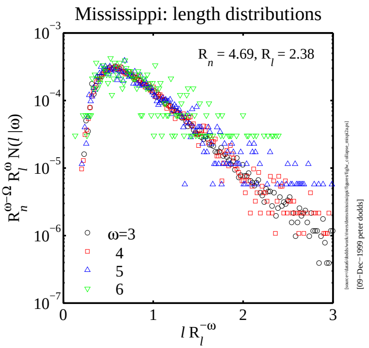

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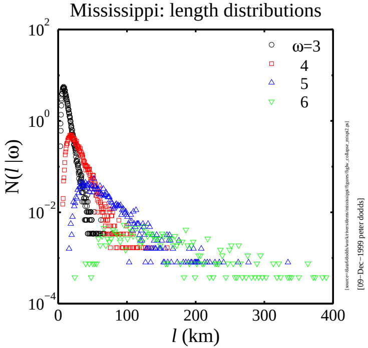

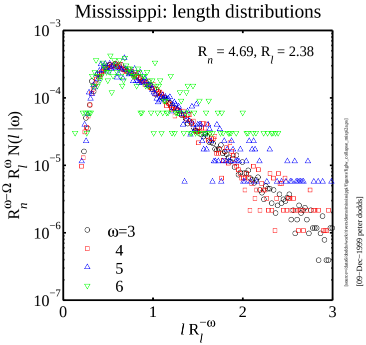

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Generalizing Horton's laws

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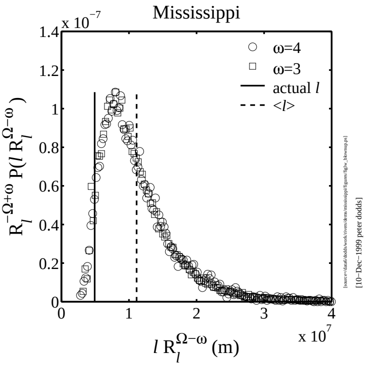

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- Predicted Mean length = 11100 km
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- Actual length/Mean length = 44 %
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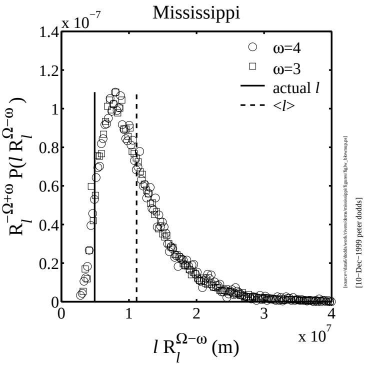

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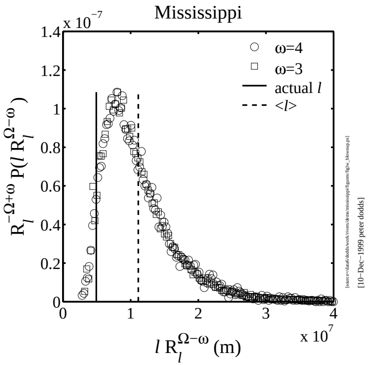

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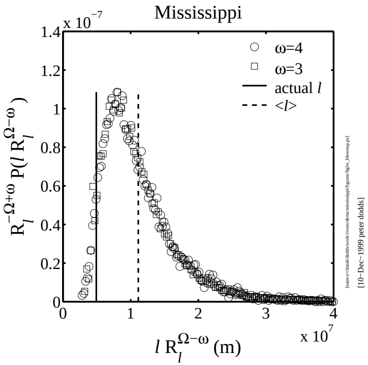
Generalizing Horton's laws

- How well does overall basin fit internal pattern?



- Actual length = 4920 km (at 1 km res)
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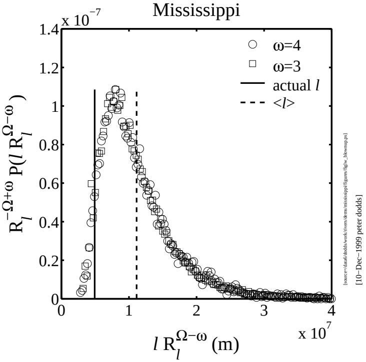

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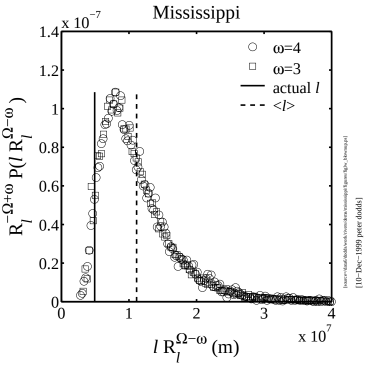

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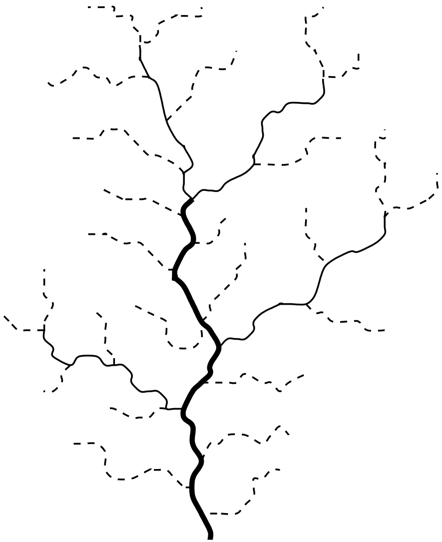
Generalizing Horton's laws

Comparison of predicted versus measured main stream lengths for large scale river networks (in 10^3 km):

basin:	ℓ_Ω	$\bar{\ell}_\Omega$	σ_ℓ	$\ell/\bar{\ell}_\Omega$	$\sigma_\ell/\bar{\ell}_\Omega$
Mississippi	4.92	11.10	5.60	0.44	0.51
Amazon	5.75	9.18	6.85	0.63	0.75
Nile	6.49	2.66	2.20	2.44	0.83
Congo	5.07	10.13	5.75	0.50	0.57
Kansas	1.07	2.37	1.74	0.45	0.73

	a	$\bar{a}_\Omega$	σ_a	$a/\bar{a}_\Omega$	$\sigma_a/\bar{a}_\Omega$
Mississippi	2.74	7.55	5.58	0.36	0.74
Amazon	5.40	9.07	8.04	0.60	0.89
Nile	3.08	0.96	0.79	3.19	0.82
Congo	3.70	10.09	8.28	0.37	0.82
Kansas	0.14	0.49	0.42	0.28	0.86

Combining stream segments distributions:



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- $$\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$$
- $P(\ell_\omega)$ is a convolution of distributions for the s_ω

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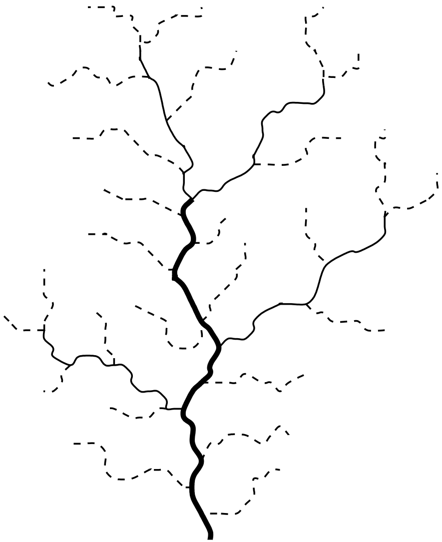

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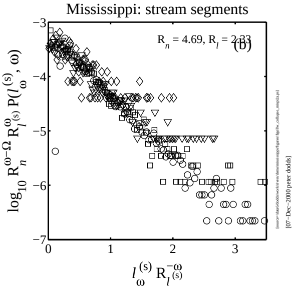
Generalizing Horton's laws

- Sum of variables $\ell_\omega = \sum_{\mu=1}^{\mu=\omega} s_\mu$ leads to convolution of distributions:

$$N(\ell|\omega) = N(s|1) * N(s|2) * \cdots * N(s|\omega)$$



$$N(s|\omega) = \frac{1}{R_n^\omega R_\ell^\omega} F\left(s/R_\ell^\omega\right)$$

$$F(x) = e^{-x/\xi}$$

Mississippi: $\xi \simeq 900$ m.

Generalizing Horton's laws

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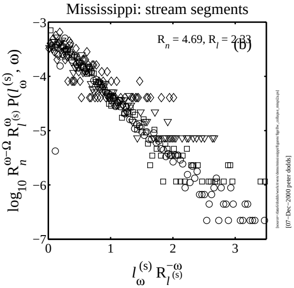

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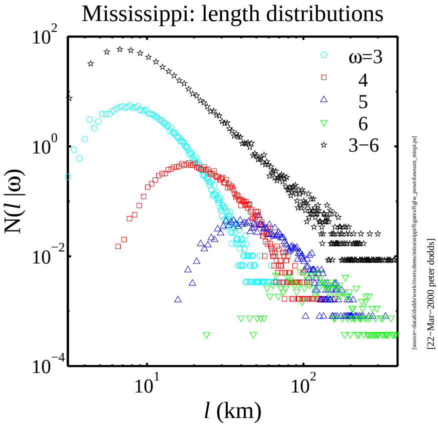

- $P(\ell) \sim \ell^{-\gamma}$
- Another round of convolutions [3]
- Interesting...

Generalizing Horton's laws

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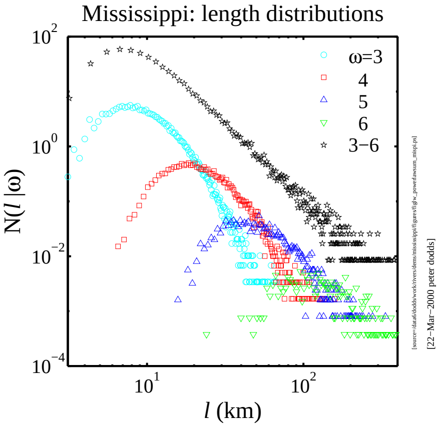

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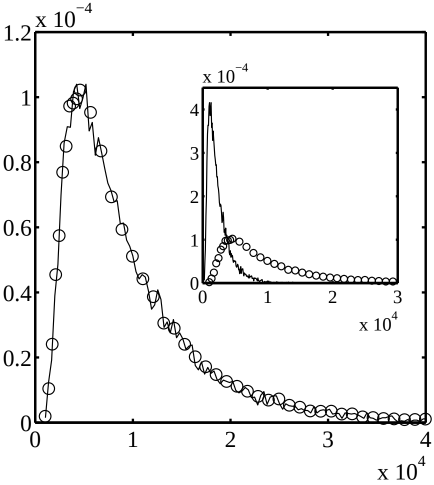
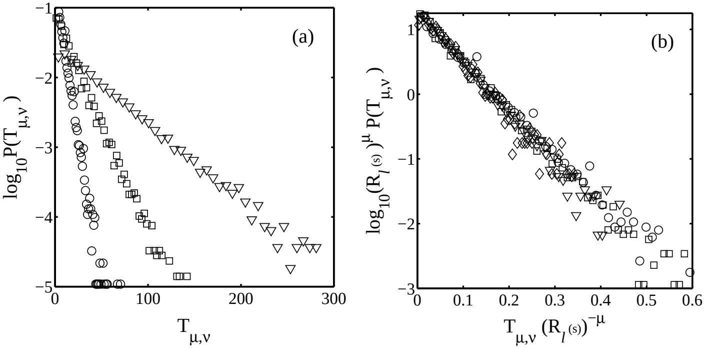
Generalizing Horton's laws

Number and area distributions for the Scheidegger model $P(n_{1,6})$ versus $P(a_6)$.



Generalizing Tokunaga's law

Scheidegger:



- Observe exponential distributions for $T_{\mu,\nu}$
- Scaling collapse works using R_s

Generalizing Tokunaga's law

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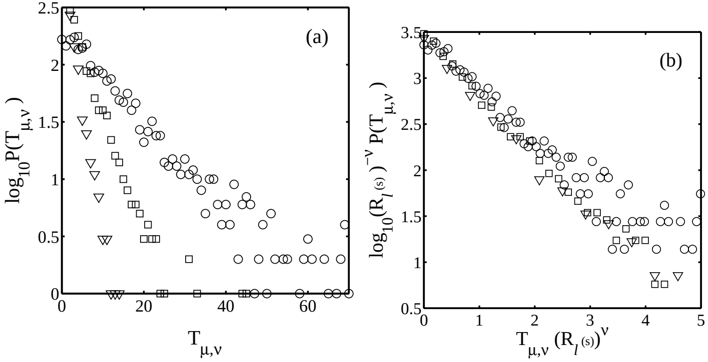


- Same data collapse for Mississippi...

Generalizing Tokunaga's law

So

$$P(T_{\mu,\nu}) = (R_s)^{\mu-\nu-1} P_t \left[T_{\mu,\nu} / (R_s)^{\mu-\nu-1} \right]$$

where

$$P_t(z) = \frac{1}{\xi_t} e^{-z/\xi_t}.$$

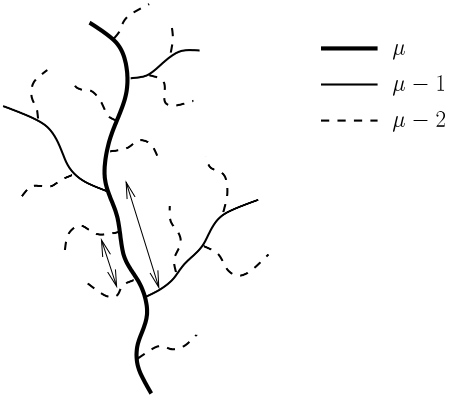
$$\boxed{P(s_\mu) \Leftrightarrow P(T_{\mu,\nu})}$$

- Exponentials arise from randomness.
- Look at joint probability $P(s_\mu, T_{\mu,\nu})$.

Generalizing Tokunaga's law

Network architecture:

- Inter-tributary lengths exponentially distributed
- Leads to random spatial distribution of stream segments



Generalizing Tokunaga's law

- Follow streams segments down stream from their beginning
- Probability (or rate) of an order μ stream segment terminating is constant:

$$\tilde{p}_\mu \simeq 1/(R_s)^{\mu-1}\xi_s$$

- Probability decays exponentially with stream order
- Inter-tributary lengths exponentially distributed
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Generalizing Tokunaga's law

- Joint distribution for generalized version of Tokunaga's law:

$$P(s_\mu, T_{\mu,\nu}) = \tilde{p}_\mu \binom{s_\mu - 1}{T_{\mu,\nu}} p_\nu^{T_{\mu,\nu}} (1 - p_\nu - \tilde{p}_\mu)^{s_\mu - T_{\mu,\nu} - 1}$$

  where
  - p_ν = probability of absorbing an order ν side stream
  - $\tilde{p}_\mu$ = probability of an order μ stream terminating
- Approximation: depends on distance units of s_μ
- In each unit of distance along stream, there is one chance of a side stream entering or the stream terminating.

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- Now deal with thing:

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- Set $(x, y) = (s_\mu, T_{\mu,\nu})$ and $q = 1 - p_\nu - \tilde{p}_\mu$, approximate liberally.
- Obtain

$$P(x, y) = Nx^{-1/2} \left[F(y/x)\right]^x$$

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$$F(v) = \left(\frac{1-v}{q}\right)^{-(1-v)} \left(\frac{v}{p}\right)^{-v}.$$

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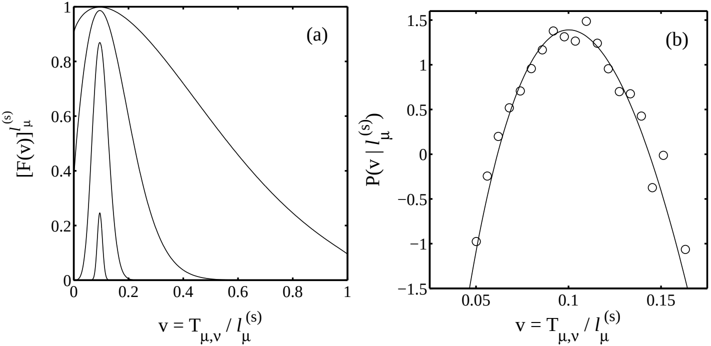
Generalizing Tokunaga's law

- Checking form of $P(s_\mu, T_{\mu,\nu})$ works:

Scheidegger:



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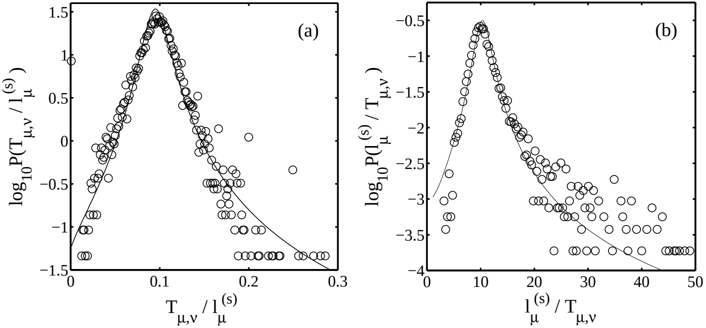

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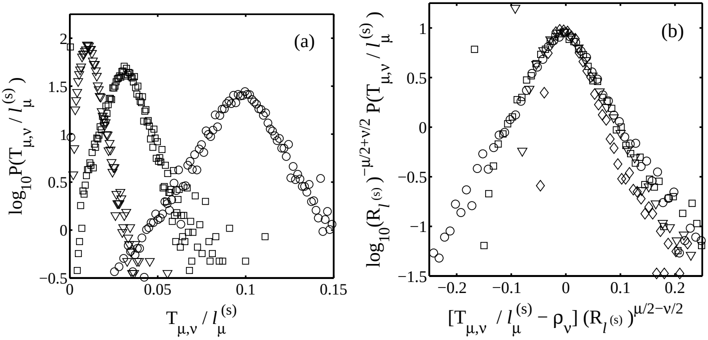

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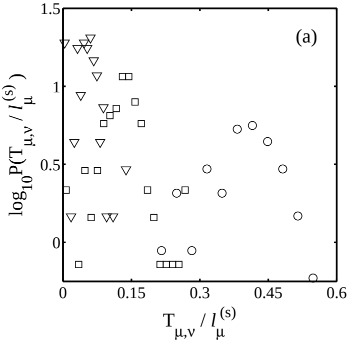
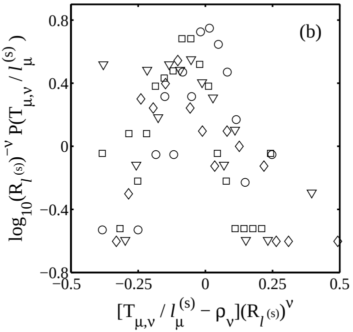
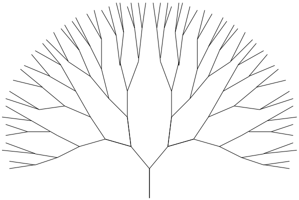




Random subnetworks on a Bethe lattice [13]



- Dominant theoretical concept for several decades.
- Bethe lattices are fun and tractable.
- Led to idea of “Statistical inevitability” of river network statistics [7]
- But Bethe lattices unconnected with surfaces.
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- So let’s move on...

Models

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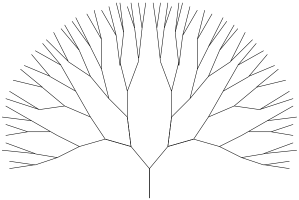

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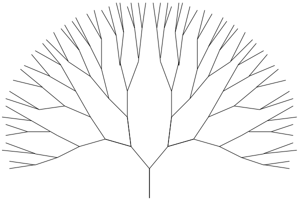

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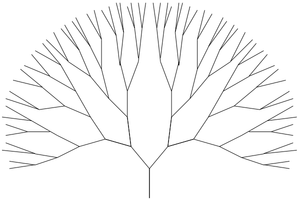

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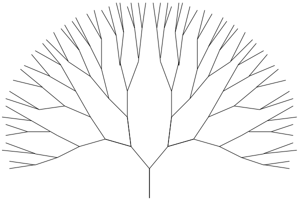

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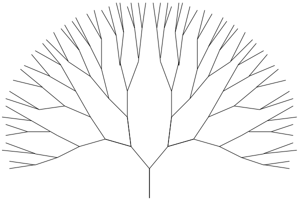

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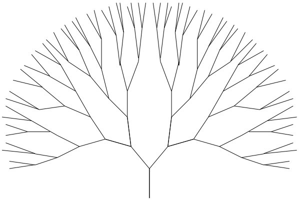

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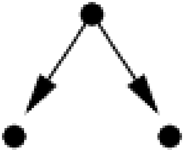
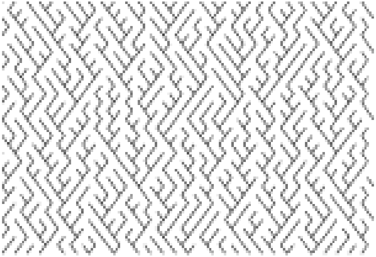
Scheidegger's model

Directed random networks [11, 12]





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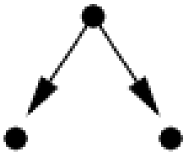

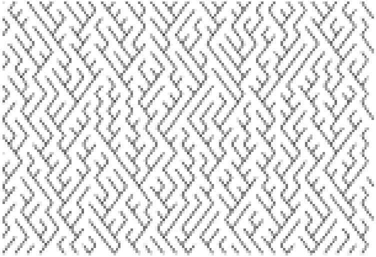

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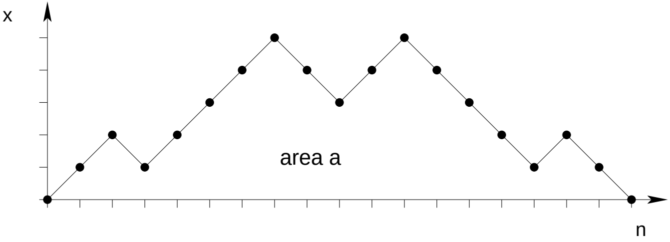
A toy model—Scheidegger's model

Random walk basins:

- Boundaries of basins are random walks



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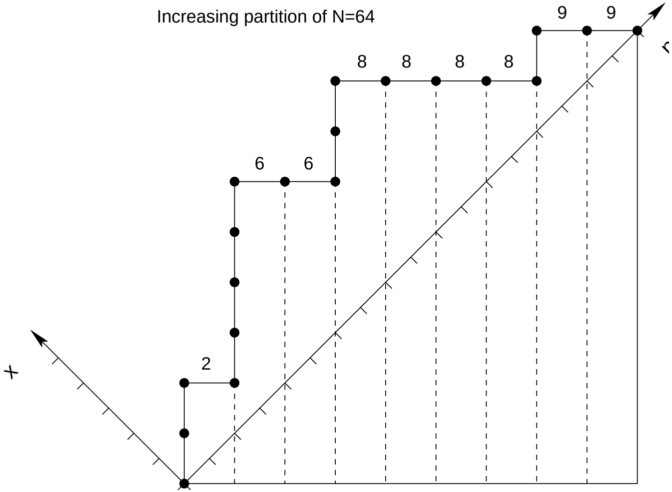

Scheidegger's model

Prob for first return of a random walk in (1+1) dimensions (from CSYS/MATH 300):

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$$P(n) \sim \frac{1}{2\sqrt{\pi}}\, n^{-3/2}.$$

  and so $P(\ell) \propto \ell^{-3/2}$.
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Scheidegger's model

Prob for first return of a random walk in (1+1) dimensions (from CSYS/MATH 300):

- $$P(n) \sim \frac{1}{2\sqrt{\pi}}\, n^{-3/2}.$$

  and so $P(\ell) \propto \ell^{-3/2}$.
- Typical area for a walk of length n is $\propto n^{3/2}$:

  $$\ell \propto a^{2/3}.$$

- Find $\tau = 4/3$, $h = 2/3$, $\gamma = 3/2$, $d = 1$.
- Note $\tau = 2 - h$ and $\gamma = 1/h$.
- R_n and R_ℓ have not been derived analytically.

Optimal channel networks

Rodríguez-Iturbe, Rinaldo, et al. [10]

- Landscapes $h(\vec{x})$ evolve such that energy dissipation $\dot{\varepsilon}$ is minimized, where

$$\dot{\varepsilon} \propto \int \mathrm{d}\vec{r}\,(\text{flux}) \times (\text{force}) \sim \sum_i a_i \nabla h_i \sim \sum_i a_i^{\gamma}$$

- Landscapes obtained numerically give exponents near that of real networks.
- But: numerical method used matters.
- And: Maritan et al. find basic universality classes are that of Scheidegger, self-similar, and a third kind of random network [8]

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Theoretical networks

Summary of universality classes:

network	h	d
Non-convergent flow	1	1
Directed random	2/3	1
Undirected random	5/8	5/4
Self-similar	1/2	1
OCN's (I)	1/2	1
OCN's (II)	2/3	1
OCN's (III)	3/5	1
Real rivers	0.5–0.7	1.0–1.2

$h \Rightarrow \ell \propto a^h$ (Hack's law).

$d \Rightarrow \ell \propto L_{\parallel}^{d}$ (stream self-affinity).

Nutshell

Branching networks II Key Points:

- Horton's laws and Tokunaga law all fit together.
- nb. for 2-d networks, these laws are 'planform' laws and ignore slope.
- Abundant scaling relations can be derived.
- Can take R_n, R_ℓ, and d as three independent parameters necessary to describe all 2-d branching networks.
- For scaling laws, only $h = \ln R_\ell / \ln R_n$ and d are needed.
- Laws can be extended nicely to laws of distributions.
- Numerous models of branching network evolution exist: nothing rock solid yet.

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