Complex Networks, CSYS/MATH 303—Assignment 3
University of Vermont, Spring 2010

Dispersed: Saturday, February 13, 2010.
Due: By start of lecture, 10:00 am, Tuesday, February 23, 2010.
Some useful reminders:
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Office hours: 1:00 pm to 2:30 pm, Wednesday @ Farrell, and by appointment
Course website: `http://www.uvm.edu/~pdodds/teaching/courses/2010-01UVM-303/`

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

Graduate students are requested to use LaTeX (or related variant).

Supply networks and allometry:

1. From lectures on Supply Networks:

   Show that for large V and $0 < \epsilon < 1/2$

   $$\min V_{\rm net} \propto \int_{\Omega_{d,D}(V)} \rho \, ||\vec{x}||^{1-2\epsilon} \, \mathrm{d}\vec{x} \sim \rho V^{1+\gamma_{\rm max}(1-2\epsilon)}$$

   Reminders: we defined $L_i = c_i^{-1} V^{\gamma_i}$ where $\gamma_1 + \gamma_2 + \ldots + \gamma_d = 1$, $\gamma_1 = \gamma_{\rm max} \geq \gamma_2 \geq \ldots \geq \gamma_d$., and $c = \prod_i c_i \leq 1$ is a shape factor.

   Hints: assume the first k lengths scale in the same way with $\gamma_1 = \ldots = \gamma_k = \gamma_{\rm max}$, and write $||\vec{x}|| = (x_1^2 + x_2^2 + \ldots + x_d^2)^{1/2}$.

2. Consider a set of rectangular areas with side lengths L_1 and L_2 such that $L_1 \propto A^{\gamma_1}$ and $L_2 \propto A^{\gamma_2}$ where A is area and $\gamma_1 + \gamma_2 = 1$. Assume $\gamma_1 > \gamma_2$ and that $\epsilon = 0$.

   Now imagine that material has to be distributed from a central source in each of these areas to sinks distributed with density $\rho(A)$, and that these sinks draw the same amount of material per unit time independent of L_1 and L_2.

   Find an exact form for how the volume of the most efficient distribution network scales with overall area $A = L_1 L_2$. (Hint: you will have to set up a double integration over the rectangle.)

   If network volume must remain a constant fraction of overall area, determine the maximal scaling of sink density ρ with A.

3. (a) For a family of d-dimensional regions, with scaling as per Question 1, determine, to leading order, the scaling of hyper-surface area S with volume V. In other words, find the exponent β in $S \propto V^{\beta}$ as $V \to \infty$.

   Assume that nothing peculiar happens with the shapes (as we have always implicitly done), in that there is no fractal roughening.

   Hint: figure out how the circumference for the rectangles in the previous question scales with area A. For d dimensions, think about how the hyper-surface area of a hyperrectangle (or orthotope) would scale.

   (b) For $d = 3$, what is the maximum possible value of β and for what values of the γ_i does this occur?