More Mechanisms for Generating Power-Law Distributions

Principles of Complex Systems
Course CSYS/MATH 300, Fall, 2009

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University of Vermont

Outline

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Another approach

Benoit Mandelbrot

- Mandelbrot = father of fractals
- Mandelbrot = almond bread
- Derived Zipf's law through optimization [11]
- Idea: Language is efficient
- Communicate as much information as possible for as little cost
- Need measures of information (H) and cost (C)...
- Minimize C/H by varying word frequency
- Recurring theme: what role does optimization play in complex systems?

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Mandelbrot vs. Simon:

- Mandelbrot (1953): “An Informational Theory of the Statistical Structure of Languages”[11]
- Simon (1955): “On a class of skew distribution functions”[14]
- Mandelbrot (1959): “A note on a class of skew distribution function: analysis and critique of a paper by H.A. Simon”
- Simon (1960): “Some further notes on a class of skew distribution functions”

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Outline

Zipfarama via Optimization

Mandelbrot's Assumptions

- Language contains n words: $w_1, w_2, \ldots, w_n$.
- ith word appears with probability p_i
- Words appear randomly according to this distribution (obviously not true...)
- Words = composition of letters is important
- Alphabet contains m letters
- Words are ordered by length (shortest first)

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Word Cost

- Length of word (plus a space)
- Word length was irrelevant for Simon's method

Objection

- Real words don't use all letter sequences

Objections to Objection

- Maybe real words roughly follow this pattern (?)
- Words can be encoded this way
- Na na na-na naaaaa...

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Binary alphabet plus a space symbol

i	1	2	3	4	5	6	7	8
word	1	10	11	100	101	110	111	1000
length	1	2	2	3	3	3	3	4
$1 + \ln_2 i$	1	2	2.58	3	3.32	3.58	3.81	4

- Word length of 2^kth word: $= k + 1$
- Word length of ith word $\simeq 1 + \log_2 i$
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Total Cost C

- Cost of the ith word: $C_i \simeq 1 + \log_m i$
- Cost of the ith word plus space: $C_i \simeq 1 + \log_m(i+1)$
- Subtract fixed cost: $C'_i = C_i - 1 \simeq \log_m(i+1)$
- Simplify base of logarithm:

$$C'_i \simeq \log_m(i+1) = \frac{\log_e(i+1)}{\log_e m}$$

- Total Cost:

$$C \sim \sum_{i=1}^{n} p_i C'_i \propto \sum_{i=1}^{n} p_i \ln(i+1)$$

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Information Measure

- Use Shannon's Entropy (or Uncertainty):

$$H = -\sum_{i=1}^{n} p_i \log_2 p_i$$

- (allegedly) von Neumann suggested 'entropy'...
- Proportional to average number of bits needed to encode each 'word' based on frequency of occurrence
- $-\log_2 p_i = \log_2 1/p_i$ = minimum number of bits needed to distinguish event i from all others
- If $p_i = 1/2$, need only 1 bit ($log_2 1/p_i = 1$)
- If $p_i = 1/64$, need 6 bits ($log_2 1/p_i = 6$)

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where $g = 1/\ln 2$

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Zipfarama via Optimization

- Minimize

$$F(p_1, p_2, \ldots, p_n) = C/H$$

subject to constraint

$$\sum_{i=1}^{n} p_i = 1$$

- Tension:
(1) Shorter words are cheaper

- (Good) question: how much does choice of C/H as function to minimize affect things?

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$$\sum_{i=1}^{n} p_i = 1$$
- Tension:
(1) Shorter words are cheaper
(2) Longer words are more informative (rarer)
- (Good) question: how much does choice of C/H as function to minimize affect things?

Zipfarama via Optimization

- Minimize

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Time for Lagrange Multipliers:

- Minimize

$$\Psi(p_1, p_2, \ldots, p_n) =$$
$$F(p_1, p_2, \ldots, p_n) + \lambda G(p_1, p_2, \ldots, p_n)$$

where

$$F(p_1, p_2, \ldots, p_n) = \frac{C}{H} = \frac{\sum_{i=1}^{n} p_i \ln(i+1)}{-g \sum_{i=1}^{n} p_i \ln p_i}$$

and the constraint function is

$$G(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} p_i - 1 = 0$$

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Insert question 4, assignment 2 (⊞)

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Some mild suffering leads to:

- $$p_j = e^{-1-\lambda H^2/gC}(j+1)^{-H/gC} \propto (j+1)^{-H/gC}$$
- A power law appears [applause]: $\alpha = H/gC$
- Next: sneakily deduce λ in terms of g, C, and H.
- Find
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Zipfarama via Optimization

Finding the exponent

- Now use the normalization constraint:

$$1 = \sum_{j=1}^{n} p_j = \sum_{j=1}^{n} (j+1)^{-H/gC} = \sum_{j=1}^{n} (j+1)^{-\alpha}$$

- As $n \to \infty$, we end up with $\zeta(H/gC) = 2$ where ζ is the Riemann Zeta Function
- Gives $\alpha \simeq 1.73$ (> 1, too high)
- If cost function changes ($j + 1 \to j + a$) then exponent is tunable
- Increase a, decrease α

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All told:

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- But optimization can involve many incommensurate elements: monetary cost, robustness, happiness,...
- Mandelbrot's argument is not super convincing
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Reconciling Mandelbrot and Simon

- Mixture of local optimization and randomness
- Numerous efforts...

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Other mechanisms:

Much argument about whether or not monkeys typing could produce Zipf's law... (Miller, 1957) [12]

Outline

Others are also not happy

Krugman and Simon

- "The Self-Organizing Economy" (Paul Krugman, 1995) [10]
- Krugman touts Zipf's law for cities, Simon's model
- "Déjà vu, Mr. Krugman" (Berry, 1999)
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Who needs a hug?

From Berry [4]

- Déjà vu, Mr. Krugman. Been there, done that. The Simon-Ijiri model was introduced to geographers in 1958 as an explanation of city size distributions, the first of many such contributions dealing with the steady states of random growth processes, ...
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- Many complex systems are prone to cascading catastrophic failure:
  - Blackouts
  - Disease outbreaks
  - Wildfires
  - Earthquakes
- But complex systems also show persistent robustness
- Robustness and Failure may be a power-law story...

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- System robustness may result from
  1. Evolutionary processes
  2. Engineering/Design
- Idea: Explore systems optimized to perform under uncertain conditions.
- The handle: 'Highly Optimized Tolerance' (HOT) [5, 6, 15]
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Features of HOT systems: [6]

- High performance and robustness
- Designed/evolved to handle known stochastic environmental variability
- Fragile in the face of unpredicted environmental signals
- Highly specialized, low entropy configurations
- Power-law distributions appear (of course...)

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HOT combines things we've seen:

- Variable transformation
- Constrained optimization

- Need power law transformation between variables: $(Y = X^{-\alpha})$
- Recall PLIPLO is bad...
- MIWO is good
- X has a characteristic size but Y does not

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Forest fire example: [6]

- Square $N \times N$ grid
- Sites contain a tree with probability ρ = density
- Sites are empty with probability $1 - \rho$
- Fires start at location according to some distribution P_{ij}
- Fires spread from tree to tree (nearest neighbor only)
- Connected clusters of trees burn completely
- Empty sites block fire
- Best case scenario:
  Build firebreaks to maximize average # trees left intact

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- Build a forest by adding one tree at a time
- Test D ways of adding one tree
- D = design parameter
- Average over P_{ij} = spark probability
- $D = 1$: random addition
- $D = N^2$: test all possibilities

Measure average area of forest left untouched

- $f(c)$ = distribution of fire sizes c (= cost)
- Yield = $Y = \rho - \langle f \rangle$

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- $D = 1$: random addition
- $D = N^2$: test all possibilities

Measure average area of forest left untouched

- $f(c)$ = distribution of fire sizes c (= cost)
- Yield = $Y = \rho - \langle f \rangle$

Robustness

Forest fire example: [6]

- Build a forest by adding one tree at a time
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Specifics:

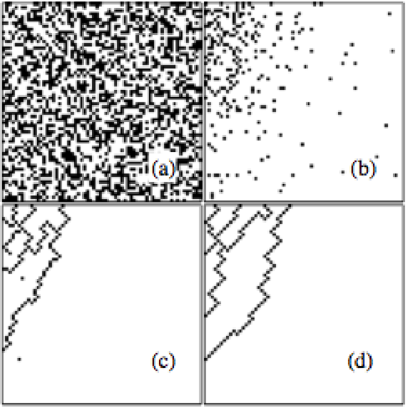
- $$P_{ij} = P_{i;a_x,b_x} P_{j;a_y,b_y}$$

  where

  $$P_{i;a,b} \propto e^{-[(i+a)/b]^2}$$

- In the original work, $b_y > b_x$
- Distribution has more width in y direction.

HOT Forests



[6]

$N = 64$

(a) $D = 1$
(b) $D = 2$
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P_{ij} has a Gaussian decay

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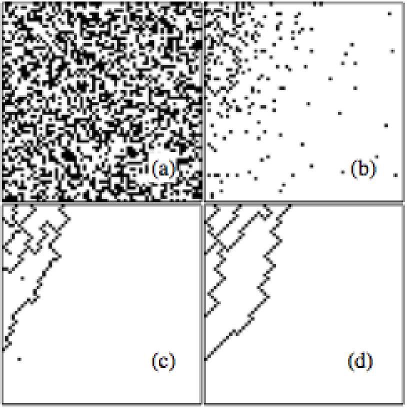

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Optimized forests do well on average

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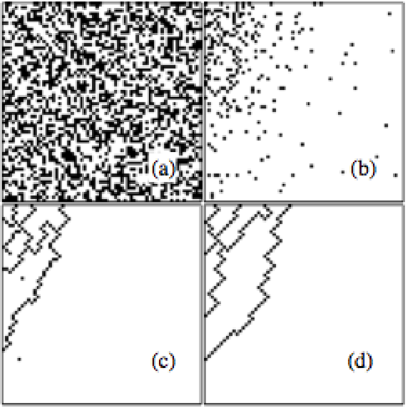

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Optimized forests do well on average but rare extreme events occur

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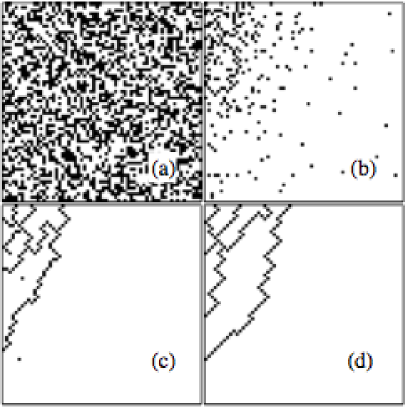

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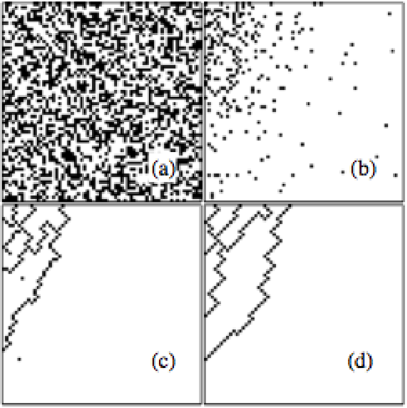

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Optimized forests do well on average (robustness) but rare extreme events occur (fragility)

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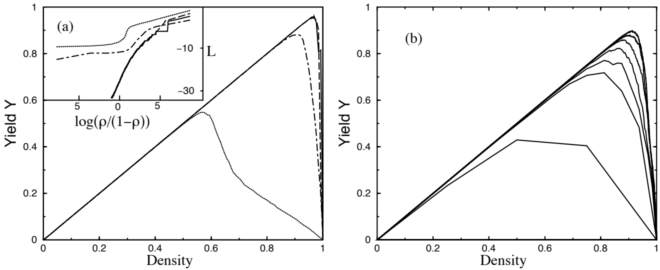

FIG. 2. Yield vs density $Y(\rho)$: (a) for design parameters $D = 1$ (dotted curve), 2 (dot-dashed), N (long dashed), and N^2 (solid) with $N = 64$, and (b) for $D = 2$ and $N = 2, 2^2, \ldots, 2^7$ running from the bottom to top curve. The results have been averaged over 100 runs. The inset to (a) illustrates corresponding loss functions $L = \log[\langle f \rangle/(1 - \langle f \rangle)]$, on a scale which more clearly differentiates between the curves.

[6]

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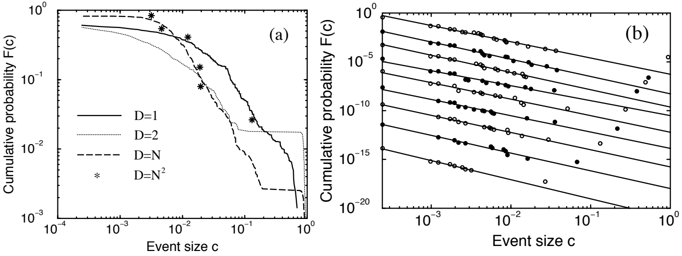

FIG. 3. Cumulative distributions of events $F(c)$: (a) at peak yield for $D = 1, 2, N$, and N^2 with $N = 64$, and (b) for $D = N^2$, and $N = 64$ at equal density increments of 0.1, ranging at $\rho = 0.1$ (bottom curve) to $\rho = 0.9$ (top curve).

[6]

Random Forests

$D = 1$: Random forests = Percolation [16]

- Randomly add trees
- Below critical density ρ_c, no fires take off
- Above critical density ρ_c, percolating cluster of trees burns
- Only at ρ_c, the critical density, is there a power-law distribution of tree cluster sizes
- Forest is random and featureless

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- Highly structured
- Power law distribution of tree cluster sizes for $\rho > \rho_c$
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HOT theory

The abstract story:

- Given $y_i = x_i^{-\alpha}$, $i = 1, \ldots, N_{\text{sites}}$
- Design system to minimize $\langle y \rangle$ subject to a constraint on the x_i
- Minimize cost:

$$C = \sum_{i=1}^{N_{\text{sites}}} Pr(y_i) y_i$$

Subject to $\sum_{i=1}^{N_{\text{sites}}} x_i = \text{constant}$

- Drag out the Lagrange Multipliers, battle away and find:

$$p_i \propto y_i^{-\gamma}$$

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HOT: Optimal fire walls in *d* dimensions

Two costs:

1. Expected size of fire

$$C_{\text{fire}} \propto \sum_{i=1}^{N_{\text{sites}}} (p_i a_i) a_i = \sum_{i=1}^{N_{\text{sites}}} p_i a_i^2$$

- a_i = area of ith site's region
- p_i = avg. prob. of fire at site in ith site's region
- N_{sites} = total number of sites

2. Cost of building and maintaining firewalls

$$C_{\text{firewalls}} \propto \sum_{i=1}^{N_{\text{sites}}} a_i^{1/2}$$

- We are assuming isometry.
- In d dimensions, 1/2 is replaced by $(d-1)/d$

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- $$0 = \frac{\partial}{\partial a_j}\left(C_{\text{fire}} - \lambda C_{\text{firewalls}}\right)$$
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Summary of designed tolerance

- Build more firewalls in areas where sparks are likely
- Small connected regions in high-danger areas
- Large connected regions in low-danger areas
- Routinely see many small outbreaks (robust)
- Rarely see large outbreaks (fragile)
- Sensitive to changes in the environment (P_{ij})

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Avalanches on Sand and Rice

SOC theory

SOC = Self-Organized Criticality

- Idea: natural dissipative systems exist at ‘critical states’
- Analogy: Ising model with temperature somehow self-tuning
- Power-law distributions of sizes and frequencies arise ‘for free’
- Introduced in 1987 by Bak, Tang, and Weisenfeld [3, 2, 9]:
  “Self-organized criticality - an explanation of 1/f noise”
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HOT versus SOC

- Both produce power laws
- Optimization versus self-tuning
- HOT systems viable over a wide range of high densities
- SOC systems have one special density
- HOT systems produce specialized structures
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Avoidance of large-scale failures

- Constrained Optimization with Limited Deviations [13]
- Weight cost of larges losses more strongly
- Increases average cluster size of burned trees...
- ... but reduces chances of catastrophe
- Power law distribution of fire sizes is truncated

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- Increases average cluster size of burned trees...
- ... but reduces chances of catastrophe
- Power law distribution of fire sizes is truncated

Cutoffs

Aside:

- Power law distributions often have an exponential cutoff

$$P(x) \sim x^{-\gamma} e^{-x/x_c}$$

where x_c is the approximate cutoff scale.

- May be stretched exponentials:

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Outline

Robustness

And we've already seen this...

- network robustness.
- Albert et al., Nature, 2000:
  "Error and attack tolerance of complex networks" [1]
- Similar robust-yet-fragile story...
- See Networks Overview, Frame 57 (⊞)

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