Mechanisms for Generating Power-Law Distributions

Principles of Complex Systems
Course CSYS/MATH 300, Fall, 2009

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Outline

Mechanisms

A powerful theme in complex systems:

- structure arises out of randomness.

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- Exhibit A: Random walks... (⊞)

Random walks

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

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$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

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Variances sum: (⊞)*

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$$= \sum_{i=1}^{t} \mathrm{Var}\left(\epsilon_i\right) = \sum_{i=1}^{t} 1 = t$$

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Random walks are weirder than you might think...

For example:

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
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- If you are behind early on, what are the chances you will make a comeback?
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See Feller, [3] Intro to Probability Theory, Volume I

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In fact:

$$\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$$

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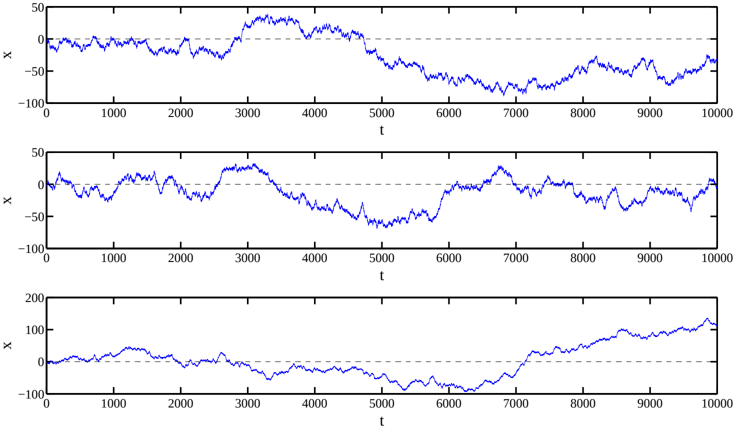
Even crazier:

The expected time between tied scores = ∞!

Random walks—some examples



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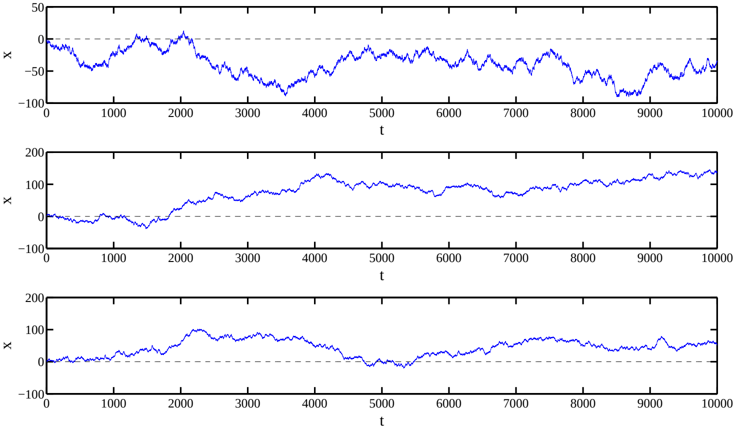

Random walks

The problem of first return:

- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

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Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent
2. Some physical structures may result from random walks
3. We'll start to see how different scalings relate to each other

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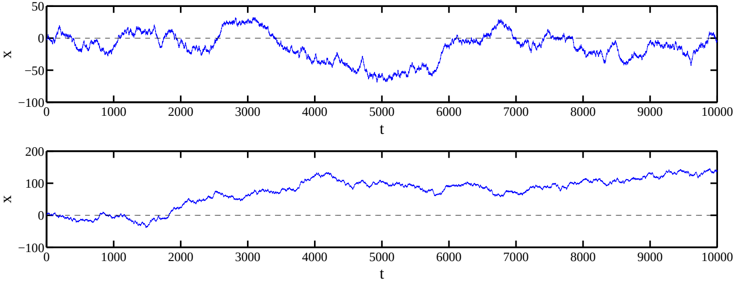

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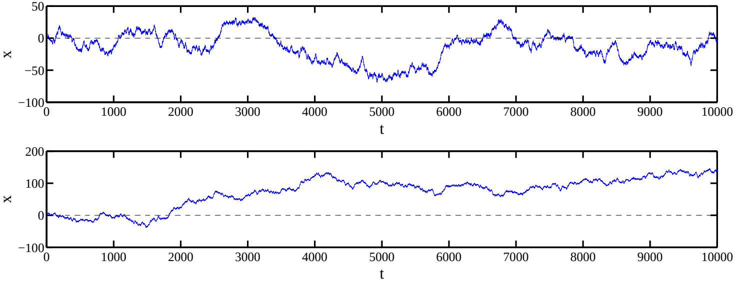
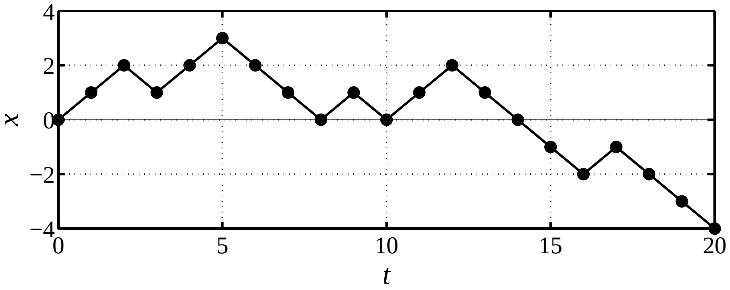


Again: expected time between ties = ∞...
Let's find out why... [3]

First Returns



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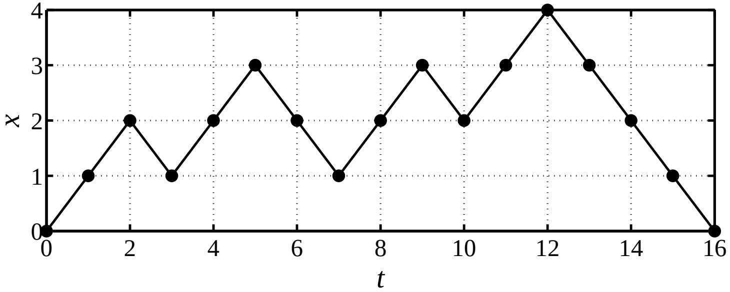
- The problem

$$P_{\text{fr}}(2n) = 2Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_{2n} = 0)$$

First Returns



- A useful restatement: $P_{\text{fr}}(2n) = 2 \cdot \frac{1}{2} Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$
- Want walks that can return many times to $x = 1$.
- (The $\frac{1}{2}$ accounts for stepping to 2 instead of 0 at $t = 2n$.)

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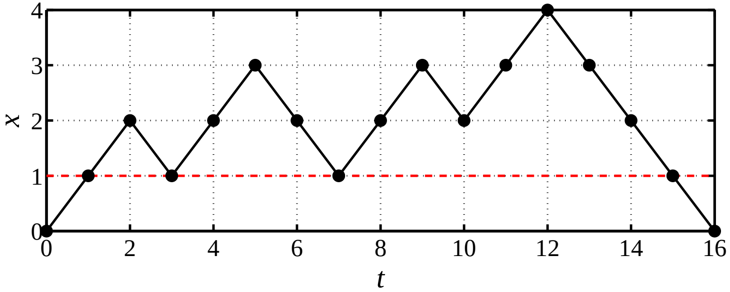

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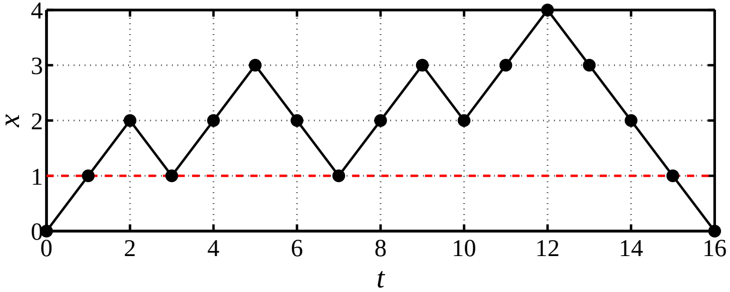

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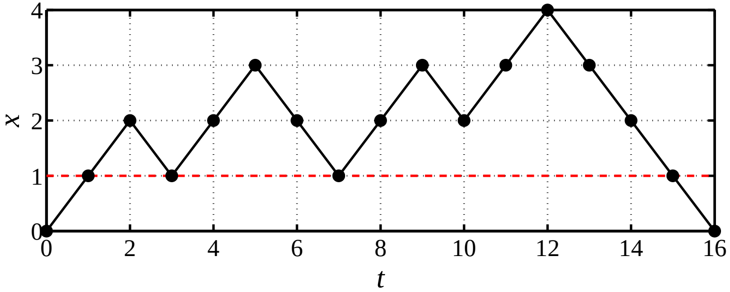

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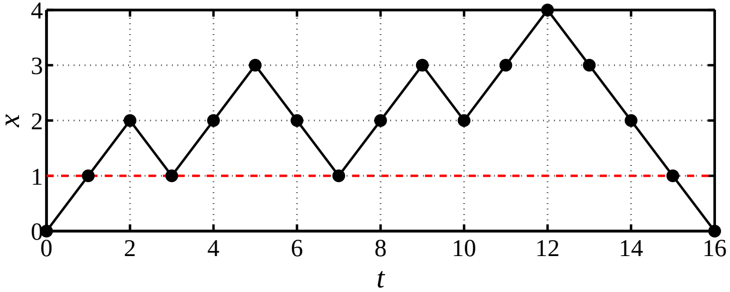

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- Counting problem (combinatorics/statistical mechanics)
- Use a method of images
- Define $N(i, j, t)$ as the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
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See this 1-1 correspondence visually...

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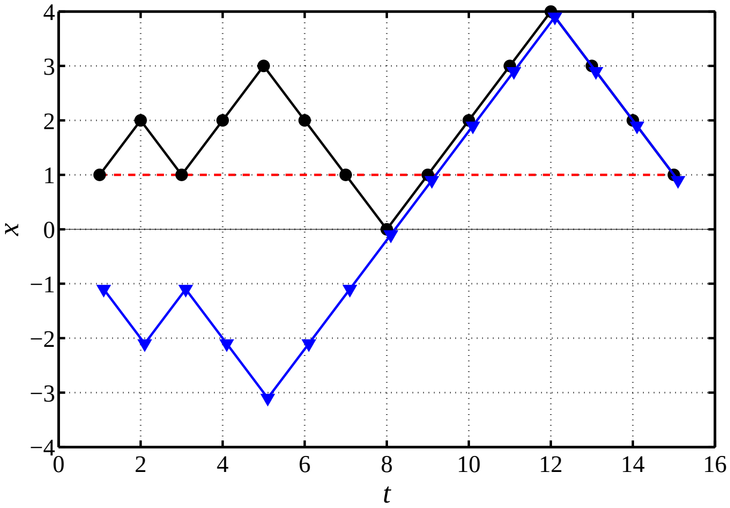

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- For any path starting at $x = 1$ that hits 0, there is a unique matching path starting at $x = -1$.
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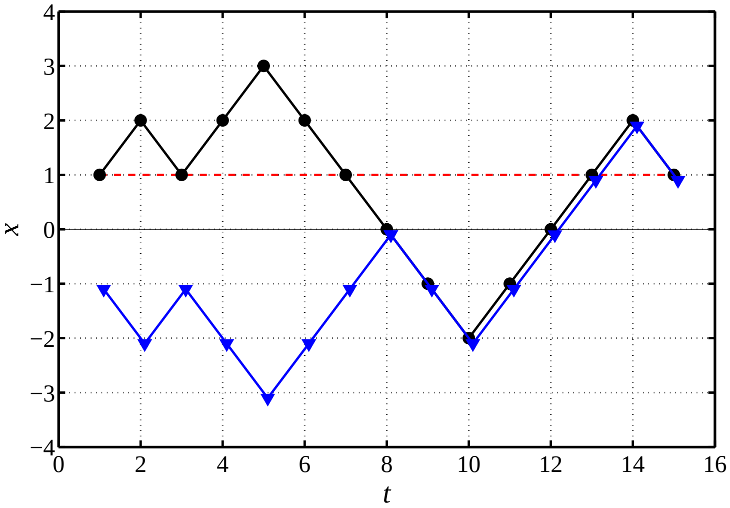

First Returns

- Next problem: what is $N(i, j, t)$?
- # positive steps + # negative steps = t.
- Random walk must displace by $j - i$ after t steps.
- # positive steps - # negative steps = $j - i$.
- # positive steps = $(t + j - i)/2$.
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We now have

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where

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Insert question 1, assignment 2 (⊞)

Find $N_{\text{first return}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$.

- Normalized Number of Paths gives Probability
- Total number of possible paths = 2^{2n}
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$$P_{\text{first return}}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$

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- $$P(t) \propto t^{-3/2},\ \gamma = 3/2$$
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Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions
- For $d = 1$, $\gamma = 3/2 \rightarrow \langle t \rangle = \infty$
- Even though walker must return, expect a long wait...

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On finite spaces:

- In any finite volume, a random walker will visit every site with equal probability
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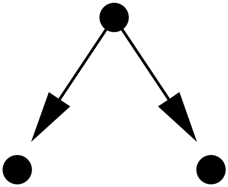
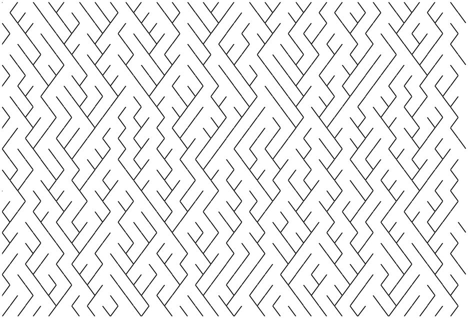
Outline

Scheidegger Networks [10, 2]





- Triangular lattice
- 'Flow' is southeast or southwest with equal probability.

Scheidegger Networks

- Creates basins with random walk boundaries
- Observe Subtracting one random walk from another gives random walk with increments

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

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Other First Returns

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- A very simple model of failure/death:
- x_t = entity's 'health' at time t
- x_0 could be > 0.
- Entity fails when x hits 0.

Streams

- Dispersion of suspended sediments in streams.
- Long times for clearing.

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1. elementary distributions (e.g., exponentials)
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- Random variable X with known distribution P_x
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- $$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$
  $$= \sum_{y|f(x)=y} P_x(f^{-1}(y)) \frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$$
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Assume relationship between x and y is 1-1.

- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
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Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

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- Exponentials arise from randomness...
- More later when we cover robustness.

Outline

Gravity

- Select a random point in space $\vec{x}$
- Measure the force of gravity $F(\vec{x})$
- Observe that $P_F(F) \sim F^{-5/2}$.

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Ingredients [12]

Matter is concentrated in stars:

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:

$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$

- Assume stars are distributed randomly in space
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:

$$F \propto r^{-2}$$

- invert:

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Transformation

- $$dF \propto d(r^{-2})$$

- $$\propto r^{-3}dr$$

- invert:

$$dr \propto r^{3}dF$$

- $$\propto F^{-3/2}dF$$

Transformation

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- $$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$
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- $$= F^{-1-3/2}\mathrm{d}F$$
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- Variance = ∞
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We need mechanisms!

Outline

Aggregation

- Random walks represent additive aggregation
- Mechanism: Random addition and subtraction
- Compare across realizations, no competition.
- Next: Random Additive/Copying Processes involving Competition.
- Widespread: Words, Cities, the Web, Wealth, Productivity (Lotka), Popularity (Books, People, ...)
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Work of Yore

- 1924: G. Udny Yule [13]:
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- 1926: Lotka [6]:
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- 1953: Mandelbrot [7]:
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Essential Extract of a Growth Model

Random Competitive Replication (RCR):

1. Start with 1 element of a particular flavor at $t = 1$
2. At time $t = 2, 3, 4, \ldots$, add a new element in one of two ways:
   - With probability ρ, create a new element with a new flavor
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Example: Words in a text

- Consider words as they appear sequentially.
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     (But mechanism for selection is not as simple...)

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- k_i = size of a group i
- $N_k(t)$ = # groups containing k elements at time t.

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Basic question: How does $N_k(t)$ evolve with time?

$$\text{First: } \sum_k k N_k(t) = t = \text{number of elements at time } t$$

Random Competitive Replication

$P_k(t)$ = Probability of choosing an element that belongs to a group of size k:

- $N_k(t)$ size k groups
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$$P_k(t) = \frac{kN_k(t)}{t}$$

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1. An element belonging to a group with k elements is replicated

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   $N_1(t+1) = N_1(t) + 1$
   Happens with probability ρ

2. A unique element is replicated.
   $N_1(t+1) = N_1(t) - 1$
   Happens with probability $(1-\rho)N_1/t$

Random Competitive Replication

Put everything together:

For $k > 1$:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho)\left((k-1)\frac{N_{k-1}(t)}{t} - k\frac{N_k(t)}{t}\right)$$

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Assume distribution stabilizes: $N_k(t) = n_k t$

(Reasonable for t large)

- Drop expectations
- Numbers of elements now fractional
- Okay over large time scales
- n_k/ρ = the fraction of groups that have size k.

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Stochastic difference equation:

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$$\Rightarrow n_k\left(1 + (1-\rho)k\right) = (1-\rho)(k-1)n_{k-1}$$

Random Competitive Replication

We have a simple recursion:

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}$$

- Interested in k large (the tail of the distribution)
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- Insert question 3, assignment 2 (⊞)

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- We (okay, you) find

$$\frac{n_k}{n_{k-1}} \simeq \left(1 - \frac{1}{k}\right)^{\frac{(2-\rho)}{(1-\rho)}}$$

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$$\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}$$

- Observe $2 < \gamma < \infty$ as ρ varies.
- For $\rho \simeq 0$ (low innovation rate):

$$\gamma \simeq 2$$

- Recalls Zipf's law: $s_r \sim r^{-\alpha}$
  (s_r = size of the rth largest element)
- We found $\alpha = 1/(\gamma - 1)$
- $\gamma = 2$ corresponds to $\alpha = 1$

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We had one other equation:

- $$\langle N_1(t+1) - N_1(t)\rangle = \rho - (1-\rho)1 \cdot \frac{N_1(t)}{t}$$
- As before, set $N_1(t) = n_1 t$ and drop expectations
- $$n_1(t+1) - n_1 t = \rho - (1-\rho)1 \cdot \frac{n_1 t}{t}$$
- $$n_1 = \rho - (1-\rho)n_1$$
- Rearrange:
  $$n_1 + (1-\rho)n_1 = \rho$$
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- Recall number of distinct elements = ρt.
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$$\frac{N_1(t)}{\rho t} = \frac{1}{2 - \rho}$$

(also = fraction of groups of size 1)

- For ρ small, fraction of unique elements $\sim 1/2$
- Roughly observed for real distributions
- ρ increases, fraction increases
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Outline

Words

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N_1 (real)	N_1 (est)	N_2 (real)	N_2 (est)
16,432	15,850	4,776	4,870

Evolution of catch phrases

- Yule's paper (1924) [13]:
  "A mathematical theory of evolution, based on the conclusions of Dr J. C. Willis, F.R.S."
- Simon's paper (1955) [11]:
  "On a class of skew distribution functions" (snore)

From Simon's introduction:

It is the purpose of this paper to analyse a class of distribution functions that appear in a wide range of empirical data—particularly data describing sociological, biological and economoic phenomena.

Its appearance is so frequent, and the phenomena so diverse, that one is led to conjecture that if these phenomena have any property in common it can only be a similarity in the structure of the underlying probability mechanisms.

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More on Herbert Simon (1916–2001):

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- Coined 'bounded rationality' and 'satisficing'
- Nearly 1000 publications
- An early leader in Artificial Intelligence, Information Processing, Decision-Making, Problem-Solving, Attention Economics, Organization Theory, Complex Systems, And Computer Simulation Of Scientific Discovery.
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Evolution of catch phrases

- Derek de Solla Price was the first to study network evolution with these kinds of models.
- Citation network of scientific papers
- Price's term: Cumulative Advantage
- Idea: papers receive new citations with probability proportional to their existing # of citations
- Directed network
- Two (surmountable) problems:
  1. New papers have no citations
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1. self-fulfilling prophecy
2. role model
3. unintended (or unanticipated) consequences
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And just to rub it in...

Merton's son, Robert C. Merton, won the Nobel Prize for Economics in 1997.

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- Another term: "Preferential Attachment"
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