Mechanisms for Generating Power-Law Distributions

Principles of Complex Systems
Course CSYS/MATH 300, Fall, 2009

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Outline

Mechanisms

A powerful theme in complex systems:

- structure arises out of randomness.
- Exhibit A: Random walks... (⊞)

Random walks

The essential random walk:

- One spatial dimension.
- Time and space are discrete
- Random walker (e.g., a drunk) starts at origin $x = 0$.
- Step at time t is ϵ_t:

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

Random walks

Displacement after t steps:

$$x_t = \sum_{i=1}^{t} \epsilon_i$$

Expected displacement:

$$\langle x_t \rangle = \left\langle \sum_{i=1}^{t} \epsilon_i \right\rangle = \sum_{i=1}^{t} \langle \epsilon_i \rangle = 0$$

Random walks

Variances sum: (⊞)*

$$\text{Var}(x_t) = \text{Var}\left(\sum_{i=1}^{t} \epsilon_i\right)$$

$$= \sum_{i=1}^{t} \text{Var}(\epsilon_i) = \sum_{i=1}^{t} 1 = t$$

* Sum rule = a good reason for using the variance to measure spread

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Random walks

So typical displacement from the origin scales as

$$\sigma = t^{1/2}$$

⇒ A non-trivial power-law arises out of additive aggregation or accumulation.

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Random walks

Random walks are weirder than you might think...

For example:

- $\xi_{r,t}$ = the probability that by time step t, a random walk has crossed the origin r times.
- Think of a coin flip game with ten thousand tosses.
- If you are behind early on, what are the chances you will make a comeback?
- The most likely number of lead changes is... 0.

See Feller, [3] Intro to Probability Theory, Volume I

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Random walks

In fact:

$$\xi_{0,t} > \xi_{1,t} > \xi_{2,t} > \cdots$$

Even crazier:

The expected time between tied scores = ∞!

Random walks—some examples

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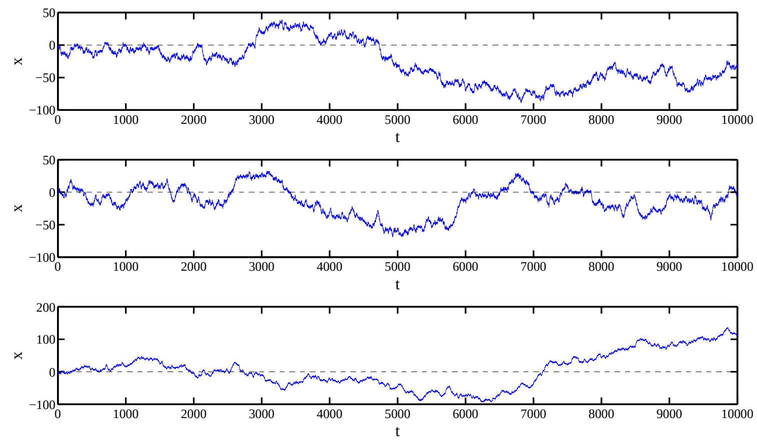

Random walks—some examples

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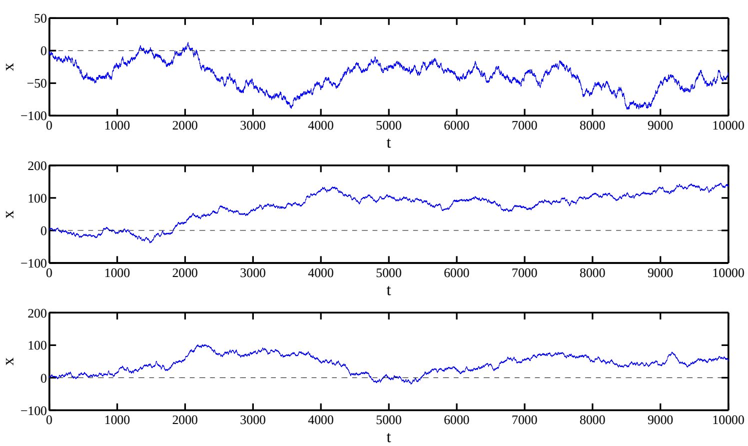

Random walks

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The problem of first return:

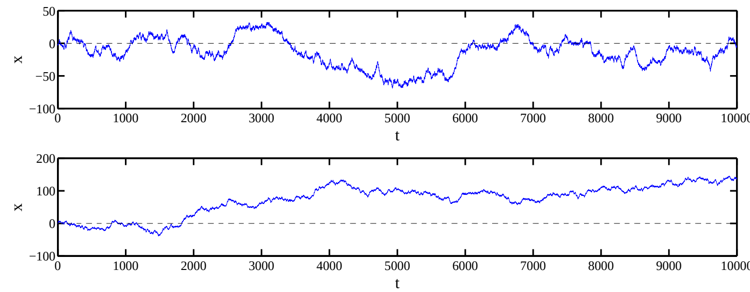
- What is the probability that a random walker in one dimension returns to the origin for the first time after t steps?
- Will our drunkard always return to the origin?
- What about higher dimensions?

First returns

Reasons for caring:

1. We will find a power-law size distribution with an interesting exponent
2. Some physical structures may result from random walks
3. We'll start to see how different scalings relate to each other

Random Walks



Again: expected time between ties = ∞...
Let's find out why... [3]

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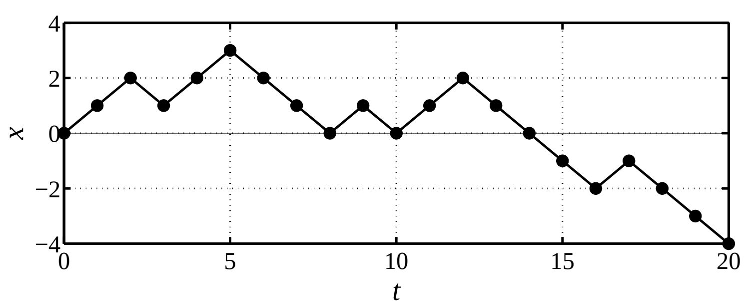
First Returns



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First Returns

For random walks in 1-d:

- Return can only happen when $t = 2n$.
- Call $P_{\text{first return}}(2n) = P_{\text{fr}}(2n)$ probability of first return at $t = 2n$.
- Assume drunkard first lurches to $x = 1$.
- The problem

$$P_{\text{fr}}(2n) = 2Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_{2n} = 0)$$

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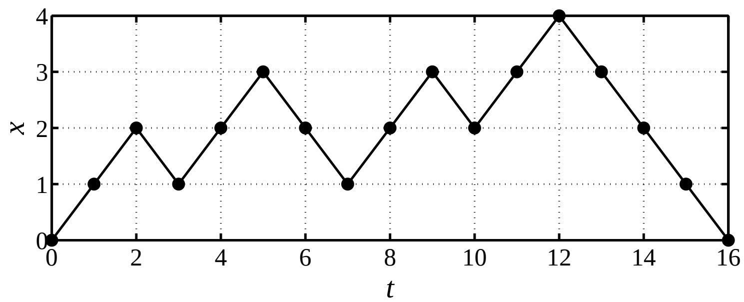
First Returns



- A useful restatement: $P_{\rm fr}(2n) = 2 \cdot \frac{1}{2} Pr(x_t \geq 1, t = 1, \ldots, 2n-1, \text{ and } x_1 = x_{2n-1} = 1)$
- Want walks that can return many times to $x = 1$.
- (The $\frac{1}{2}$ accounts for stepping to 2 instead of 0 at $t = 2n$.)

First Returns

- Counting problem (combinatorics/statistical mechanics)
- Use a method of images
- Define $N(i, j, t)$ as the # of possible walks between $x = i$ and $x = j$ taking t steps.
- Consider all paths starting at $x = 1$ and ending at $x = 1$ after $t = 2n - 2$ steps.
- Subtract how many hit $x = 0$.

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First Returns

Key observation:
of t-step paths starting and ending at $x = 1$
and hitting $x = 0$ at least once
= # of t-step paths starting at $x = -1$ and ending at $x = 1$
= $N(-1, 1, t)$

So $N_{\text{first return}}(2n) = N(1, 1, 2n-2) - N(-1, 1, 2n-2)$

See this 1-1 correspondence visually...

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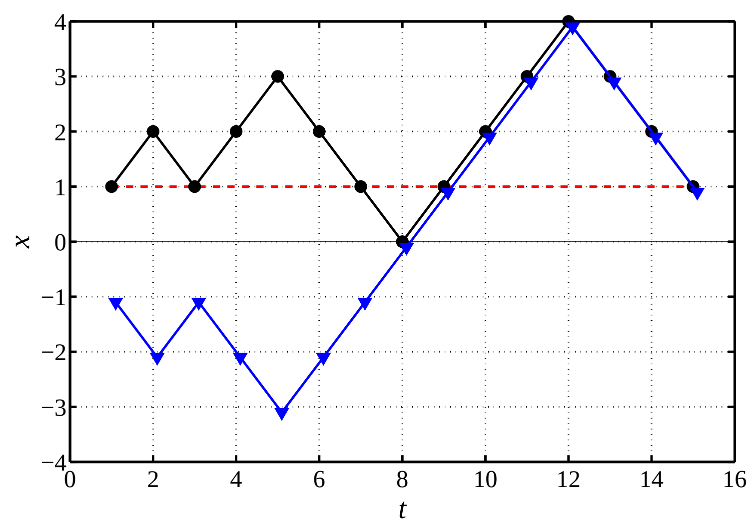

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First Returns

- For any path starting at $x = 1$ that hits 0, there is a unique matching path starting at $x = -1$.
- Matching path first mirrors and then tracks.

First Returns

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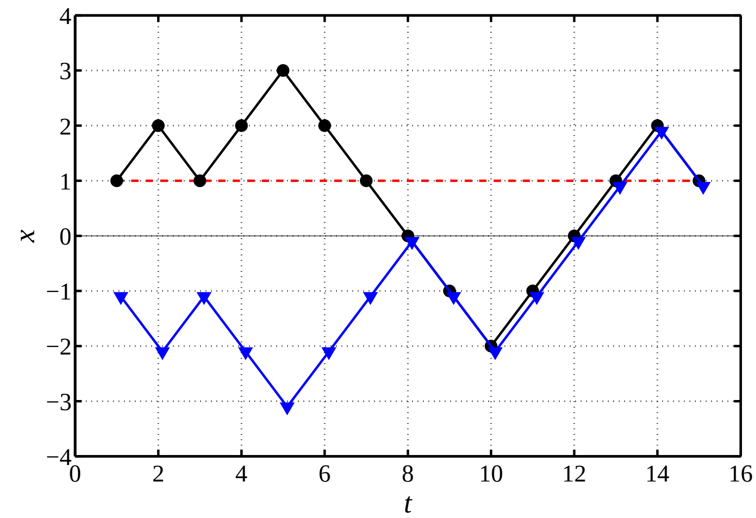

First Returns

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- Next problem: what is $N(i, j, t)$?
- # positive steps + # negative steps = t.
- Random walk must displace by $j - i$ after t steps.
- # positive steps - # negative steps = $j - i$.
- # positive steps = $(t + j - i)/2$.
- $$N(i, j, t) = \binom{t}{\#\text{ positive steps}} = \binom{t}{(t + j - i)/2}$$

First Returns

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We now have

$$N_{\text{first return}}(2n) = N(1, 1, 2n - 2) - N(-1, 1, 2n - 2)$$

where

$$N(i, j, t) = \binom{t}{(t + j - i)/2}$$

First Returns

Insert question 1, assignment 2 (⊞)

Find $N_{\text{first return}}(2n) \sim \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$.

- Normalized Number of Paths gives Probability
- Total number of possible paths = 2^{2n}
-

$$P_{\text{first return}}(2n) = \frac{1}{2^{2n}} N_{\text{first return}}(2n)$$

$$\simeq \frac{1}{2^{2n}} \frac{2^{2n-3/2}}{\sqrt{2\pi}n^{3/2}}$$

$$= \frac{1}{\sqrt{2\pi}}(2n)^{-3/2}$$

First Returns

- Same scaling holds for continuous space/time walks.
-

$$P(t) \propto t^{-3/2}, \ \gamma = 3/2$$

- $P(t)$ is normalizable
- Recurrence: Random walker always returns to origin
- Moral: Repeated gambling against an infinitely wealthy opponent must lead to ruin.

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First Returns

Higher dimensions:

- Walker in $d = 2$ dimensions must also return
- Walker may not return in $d \geq 3$ dimensions
- For $d = 1$, $\gamma = 3/2 \rightarrow \langle t \rangle = \infty$
- Even though walker must return, expect a long wait...

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Random walks

On finite spaces:

- In any finite volume, a random walker will visit every site with equal probability
- Random walking $\equiv$ Diffusion
- Call this probability the Invariant Density of a dynamical system
- Non-trivial Invariant Densities arise in chaotic systems.

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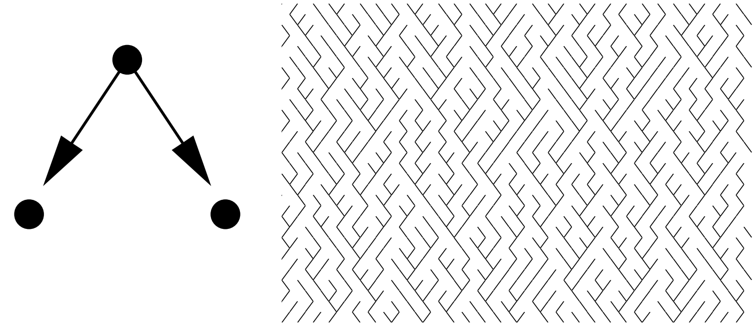
Random walks on

On networks:

- On networks, a random walker visits each node with frequency $\propto$ node degree
- Equal probability still present: walkers traverse edges with equal frequency.

Scheidegger Networks [10, 2]



- Triangular lattice
- 'Flow' is southeast or southwest with equal probability.

Scheidegger Networks

- Creates basins with random walk boundaries
- Observe Subtracting one random walk from another gives random walk with increments

$$\epsilon_t = \begin{cases} +1 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

- Basin length ℓ distribution: $P(\ell) \propto \ell^{-3/2}$

Connections between Exponents

- For a basin of length ℓ, width $\propto \ell^{1/2}$
- Basin area $a \propto \ell \cdot \ell^{1/2} = \ell^{3/2}$
- Invert: $\ell \propto a^{2/3}$
- $\mathrm{d}\ell \propto \mathrm{d}(a^{2/3}) = 2/3 a^{-1/3}\mathrm{d}a$
- $Pr(\text{basin area} = a)\mathrm{d}a$
  $= Pr(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-3/2}\mathrm{d}\ell$
  $\propto (a^{2/3})^{-3/2} a^{-1/3}\mathrm{d}a$
  $= a^{-4/3}\mathrm{d}a$
  $= a^{-\tau}\mathrm{d}a$

Connections between Exponents

- Both basin area and length obey power law distributions
- Observed for real river networks
- Typically: $1.3 < \beta < 1.5$ and $1.5 < \gamma < 2$
- Smaller basins more allometric ($h > 1/2$)
- Larger basins more isometric ($h = 1/2$)

Connections between Exponents

- Generalize relationship between area and length
- Hack's law [4]:

$$\ell \propto a^h$$

  where $0.5 \lesssim h \lesssim 0.7$
- Redo calc with γ, τ, and h.

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Connections between Exponents

- Given

$$\ell \propto a^h, \ P(a) \propto a^{-\tau}, \text{ and } P(\ell) \propto \ell^{-\gamma}$$

- $\mathrm{d}\ell \propto \mathrm{d}(a^h) = h a^{h-1} \mathrm{d}a$
- $Pr(\text{basin area} = a)\mathrm{d}a$
  $= Pr(\text{basin length} = \ell)\mathrm{d}\ell$
  $\propto \ell^{-\gamma}\mathrm{d}\ell$
  $\propto (a^h)^{-\gamma} a^{h-1}\mathrm{d}a$
  $= a^{-(1+h(\gamma-1))}\mathrm{d}a$
-

$$\boxed{\tau = 1 + h(\gamma - 1)}$$

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Connections between Exponents

With more detailed description of network structure, $\tau = 1 + h(\gamma - 1)$ simplifies:

$$\tau = 2 - h$$

$$\gamma = 1/h$$

- Only one exponent is independent
- Simplify system description
- Expect scaling relations where power laws are found
- Characterize universality class with independent exponents

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Other First Returns

Failure

- A very simple model of failure/death:
- x_t = entity's 'health' at time t
- x_0 could be > 0.
- Entity fails when x hits 0.

Streams

- Dispersion of suspended sediments in streams.
- Long times for clearing.

More than randomness

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Frame 40/88

- Can generalize to Fractional Random Walks
- Levy flights, Fractional Brownian Motion
- In 1-d,

$$\sigma \sim t^{\alpha}$$

$\alpha > 1/2$ — superdiffusive
$\alpha < 1/2$ — subdiffusive

- Extensive memory of path now matters...

Variable Transformation

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Frame 42/88

Understand power laws as arising from

1. elementary distributions (e.g., exponentials)
2. variables connected by power relationships

Variable Transformation

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Frame 43/88

- Random variable X with known distribution P_x
- Second random variable Y with $y = f(x)$.
- $$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$
$$= \sum_{y|f(x)=y} P_x(f^{-1}(y)) \frac{\mathrm{d}y}{|f'(f^{-1}(y))|}$$
- Easier to do by hand...

General Example

Assume relationship between x and y is 1-1.

- Power-law relationship between variables: $y = cx^{-\alpha}$, $\alpha > 0$
- Look at y large and x small
-

$$\mathrm{d}y = \mathrm{d}\left(cx^{-\alpha}\right)$$

$$= c(-\alpha)x^{-\alpha-1}\mathrm{d}x$$

$$\text{invert: } \mathrm{d}x = \frac{-1}{c\alpha}x^{\alpha+1}\mathrm{d}y$$

$$\mathrm{d}x = \frac{-1}{c\alpha}\left(\frac{y}{c}\right)^{-(\alpha+1)/\alpha}\mathrm{d}y$$

$$\mathrm{d}x = \frac{-c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y$$

General Example

Now make transformation:

$$P_y(y)\mathrm{d}y = P_x(x)\mathrm{d}x$$

$$P_y(y)\mathrm{d}y = P_x\overbrace{\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)}^{(x)}\overbrace{\frac{c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y}^{\mathrm{d}x}$$

- So $P_y(y) \propto y^{-1-1/\alpha}$ as $y \to \infty$ providing $P_x(x) \to$ constant as $x \to 0$.

General Example

$$P_y(y)\mathrm{d}y = P_x\left(\left(\frac{y}{c}\right)^{-1/\alpha}\right)\frac{c^{1/\alpha}}{\alpha}y^{-1-1/\alpha}\mathrm{d}y$$

- If $P_x(x) \to x^{\beta}$ as $x \to 0$ then

$$P_y(y) \propto y^{-1-1/\alpha-\beta/\alpha} \text{ as } y \to \infty$$

Example

Exponential distribution

Given $P_x(x) = \frac{1}{\lambda}e^{-x/\lambda}$ and $y = cx^{-\alpha}$, then

$$P(y) \propto y^{-1-1/\alpha} + O\left(y^{-1-2/\alpha}\right)$$

- Exponentials arise from randomness...
- More later when we cover robustness.

Gravity

- Select a random point in space $\vec{x}$
- Measure the force of gravity $F(\vec{x})$
- Observe that $P_F(F) \sim F^{-5/2}$.

Ingredients [12]

Matter is concentrated in stars:

- F is distributed unevenly
- Probability of being a distance r from a single star at $\vec{x} = \vec{0}$:
$$P_r(r)\mathrm{d}r \propto r^2 \mathrm{d}r$$
- Assume stars are distributed randomly in space
- Assume only one star has significant effect at $\vec{x}$.
- Law of gravity:
$$F \propto r^{-2}$$
- invert:
$$r \propto F^{-1/2}$$

Transformation

- $$\mathrm{d}F \propto \mathrm{d}(r^{-2})$$
- $$\propto r^{-3}\mathrm{d}r$$
- invert: $$\mathrm{d}r \propto r^3 \mathrm{d}F$$
- $$\propto F^{-3/2}\mathrm{d}F$$

Transformation

Using $\boxed{r \propto F^{-1/2}}$, $\boxed{\mathrm{d}r \propto F^{-3/2}\mathrm{d}F}$ and $\boxed{P_r(r) \propto r^2}$

- $$P_F(F)\mathrm{d}F = P_r(r)\mathrm{d}r$$
- $$\propto P_r(F^{-1/2})F^{-3/2}\mathrm{d}F$$
- $$\propto \left(F^{-1/2}\right)^2 F^{-3/2}\mathrm{d}F$$
- $$= F^{-1-3/2}\mathrm{d}F$$
- $$= F^{-5/2}\mathrm{d}F$$

Gravity

$$P_F(F) = F^{-5/2}\mathrm{d}F$$

- $$\gamma = 5/2$$
- Mean is finite
- Variance = ∞
- A wild distribution
- Random sampling of space usually safe but can end badly...

Caution!

PLIPLO = Power law in, power law out

Explain a power law as resulting from another unexplained power law.

Don't do this!!! (slap, slap)

We need mechanisms!

Aggregation

- Random walks represent additive aggregation
- Mechanism: Random addition and subtraction
- Compare across realizations, no competition.
- Next: Random Additive/Copying Processes involving Competition.
- Widespread: Words, Cities, the Web, Wealth, Productivity (Lotka), Popularity (Books, People, ...)
- Competing mechanisms (trickiness)

Work of Yore

- 1924: G. Udny Yule [13]:
  # Species per Genus
- 1926: Lotka [6]:
  # Scientific papers per author (Lotka's law)
- 1953: Mandelbrot [7]:
  Optimality argument for Zipf's law; focus on language.
- 1955: Herbert Simon [11, 14]:
  Zipf's law for word frequency, city size, income, publications, and species per genus.
- 1965/1976: Derek de Solla Price [8, 9]:
  Network of Scientific Citations.
- 1999: Barabasi and Albert [1]:
  The World Wide Web, networks-at-large.

Essential Extract of a Growth Model

Random Competitive Replication (RCR):

1. Start with 1 element of a particular flavor at $t = 1$
2. At time $t = 2, 3, 4, \ldots$, add a new element in one of two ways:
    - With probability ρ, create a new element with a new flavor
      ➤ Mutation/Innovation
    - With probability $1 - \rho$, randomly choose from all existing elements, and make a copy.
      ➤ Replication/Imitation
    - Elements of the same flavor form a group

Random Competitive Replication

Example: Words in a text

- Consider words as they appear sequentially.
- With probability ρ, the next word has not previously appeared
  ➤ Mutation/Innovation
- With probability $1 - \rho$, randomly choose one word from all words that have come before, and reuse this word
  ➤ Replication/Imitation

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Random Competitive Replication

- Competition for replication between elements is random
- Competition for growth between groups is not random
- Selection on groups is biased by size
- Rich-gets-richer story
- Random selection is easy
- No great knowledge of system needed

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Random Competitive Replication

- Steady growth of system: +1 element per unit time.
- Steady growth of distinct flavors at rate ρ
- We can incorporate
    1. Element elimination
    2. Elements moving between groups
    3. Variable innovation rate ρ
    4. Different selection based on group size
       (But mechanism for selection is not as simple...)

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Random Competitive Replication

Definitions:

- k_i = size of a group i
- $N_k(t)$ = # groups containing k elements at time t.

Basic question: How does $N_k(t)$ evolve with time?

First: $\sum_k kN_k(t) = t =$ number of elements at time t

Random Competitive Replication

$P_k(t)$ = Probability of choosing an element that belongs to a group of size k:

- $N_k(t)$ size k groups
- $\Rightarrow kN_k(t)$ elements in size k groups
- t elements overall

$$P_k(t) = \frac{kN_k(t)}{t}$$

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Random Competitive Replication

$N_k(t)$, the number of groups with k elements, changes at time t if

1. An element belonging to a group with k elements is replicated
   $N_k(t+1) = N_k(t) - 1$
   Happens with probability $(1-\rho)kN_k(t)/t$

2. An element belonging to a group with $k-1$ elements is replicated
   $N_k(t+1) = N_k(t) + 1$
   Happens with probability $(1-\rho)(k-1)N_{k-1}(t)/t$

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Random Competitive Replication

Special case for $N_1(t)$:

1. The new element is a new flavor:
   $N_1(t+1) = N_1(t) + 1$
   Happens with probability ρ

2. A unique element is replicated.
   $N_1(t+1) = N_1(t) - 1$
   Happens with probability $(1-\rho)N_1/t$

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Random Competitive Replication

Put everything together:

For $k > 1$:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho)\left((k-1)\frac{N_{k-1}(t)}{t} - k\frac{N_k(t)}{t}\right)$$

For $k = 1$:

$$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho)1 \cdot \frac{N_1(t)}{t}$$

Random Competitive Replication

Assume distribution stabilizes: $N_k(t) = n_k t$

(Reasonable for t large)

- Drop expectations
- Numbers of elements now fractional
- Okay over large time scales
- n_k/ρ = the fraction of groups that have size k.

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Random Competitive Replication

Stochastic difference equation:

$$\langle N_k(t+1) - N_k(t) \rangle = (1-\rho)\left((k-1)\frac{N_{k-1}(t)}{t} - k\frac{N_k(t)}{t}\right)$$

becomes

$$n_k(t+1) - n_k t = (1-\rho)\left((k-1)\frac{n_{k-1}t}{t} - k\frac{n_k t}{t}\right)$$

$$n_k(\cancel{t} + 1 - \cancel{t}) = (1-\rho)\left((k-1)\frac{n_{k-1}\cancel{t}}{\cancel{t}} - k\frac{n_k \cancel{t}}{\cancel{t}}\right)$$

$$\Rightarrow n_k = (1-\rho)\left((k-1)n_{k-1} - k n_k\right)$$

$$\Rightarrow n_k\left(1 + (1-\rho)k\right) = (1-\rho)(k-1)n_{k-1}$$

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Random Competitive Replication

We have a simple recursion:

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}$$

- Interested in k large (the tail of the distribution)
- Expand as a series of powers of $1/k$
- Insert question 3, assignment 2 (⊞)

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Random Competitive Replication

- We (okay, you) find

$$\frac{n_k}{n_{k-1}} \simeq (1 - \frac{1}{k})^{\frac{(2-\rho)}{(1-\rho)}}$$

- $$\frac{n_k}{n_{k-1}} \simeq \left(\frac{k-1}{k}\right)^{\frac{(2-\rho)}{(1-\rho)}}$$

- $$n_k \propto k^{-\frac{(2-\rho)}{(1-\rho)}} = k^{-\gamma}$$

$$\boxed{\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}}$$

Random Competitive Replication

$$\boxed{\gamma = \frac{(2-\rho)}{(1-\rho)} = 1 + \frac{1}{(1-\rho)}}$$

- Observe $2 < \gamma < \infty$ as ρ varies.
- For $\rho \simeq 0$ (low innovation rate):

$$\gamma \simeq 2$$

- Recalls Zipf's law: $s_r \sim r^{-\alpha}$
  (s_r = size of the rth largest element)
- We found $\alpha = 1/(\gamma - 1)$
- $\gamma = 2$ corresponds to $\alpha = 1$

Random Competitive Replication

- We (roughly) see Zipfian exponent [14] of $\alpha = 1$ for many real systems: city sizes, word distributions, ...
- Corresponds to $\rho \rightarrow 0$ (Krugman doesn't like it) [5]
- But still other mechanisms are possible...
- Must look at the details to see if mechanism makes sense...

Random Competitive Replication

We had one other equation:

- $$\langle N_1(t+1) - N_1(t) \rangle = \rho - (1-\rho) 1 \cdot \frac{N_1(t)}{t}$$
- As before, set $N_1(t) = n_1 t$ and drop expectations
- $$n_1(t+1) - n_1 t = \rho - (1-\rho) 1 \cdot \frac{n_1 t}{t}$$
- $$n_1 = \rho - (1-\rho) n_1$$
- Rearrange:

$$n_1 + (1-\rho) n_1 = \rho$$

- $$\boxed{n_1 = \frac{\rho}{2-\rho}}$$

Random Competitive Replication

So... $N_1(t) = n_1 t = \frac{\rho t}{2 - \rho}$

- Recall number of distinct elements = ρt.
- Fraction of distinct elements that are unique (belong to groups of size 1):

$$\frac{N_1(t)}{\rho t} = \frac{1}{2 - \rho}$$

  (also = fraction of groups of size 1)
- For ρ small, fraction of unique elements $\sim 1/2$
- Roughly observed for real distributions
- ρ increases, fraction increases
- Can show fraction of groups with two elements $\sim 1/6$
- Model does well at both ends of the distribution

Words

From Simon [11]:

Estimate $\rho_{\text{est}} =$ # unique words/# all words

For Joyce's Ulysses: $\rho_{\text{est}} \simeq 0.115$

N_1 (real)	N_1 (est)	N_2 (real)	N_2 (est)
16,432	15,850	4,776	4,870

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Evolution of catch phrases

- Yule's paper (1924) [13]:
  "A mathematical theory of evolution, based on the conclusions of Dr J. C. Willis, F.R.S."
- Simon's paper (1955) [11]:
  "On a class of skew distribution functions" (snore)

From Simon's introduction:

It is the purpose of this paper to analyse a class of distribution functions that appear in a wide range of empirical data—particularly data describing sociological, biological and economic phenomena.

Its appearance is so frequent, and the phenomena so diverse, that one is led to conjecture that if these phenomena have any property in common it can only be a similarity in the structure of the underlying probability mechanisms.

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Evolution of catch phrases

More on Herbert Simon (1916–2001):

- Political scientist
- Involved in Cognitive Psychology, Computer Science, Public Administration, Economics, Management, Sociology
- Coined 'bounded rationality' and 'satisficing'
- Nearly 1000 publications
- An early leader in Artificial Intelligence, Information Processing, Decision-Making, Problem-Solving, Attention Economics, Organization Theory, Complex Systems, And Computer Simulation Of Scientific Discovery.
- Nobel Laureate in Economics

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Evolution of catch phrases

- Derek de Solla Price was the first to study network evolution with these kinds of models.
- Citation network of scientific papers
- Price's term: Cumulative Advantage
- Idea: papers receive new citations with probability proportional to their existing # of citations
- Directed network
- Two (surmountable) problems:
    1. New papers have no citations
    2. Selection mechanism is more complicated

Evolution of catch phrases

- Robert K. Merton: the Matthew Effect
- Studied careers of scientists and found credit flowed disproportionately to the already famous

  From the Gospel of Matthew:
  "For to every one that hath shall be given...
  (Wait! There's more....)
  but from him that hath not, that also which he seemeth to have shall be taken away.
  And cast the worthless servant into the outer darkness; there men will weep and gnash their teeth."
- Matilda effect: women's scientific achievements are often overlooked

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Evolution of catch phrases

Merton was a catchphrase machine:

1. self-fulfilling prophecy
2. role model
3. unintended (or unanticipated) consequences
4. focused interview → focus group

And just to rub it in...

Merton's son, Robert C. Merton, won the Nobel Prize for Economics in 1997.

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Evolution of catch phrases

- Barabasi and Albert [1]—thinking about the Web
- Independent reinvention of a version of Simon and Price's theory for networks
- Another term: "Preferential Attachment"
- Considered undirected networks (not realistic but avoids 0 citation problem)
- Still have selection problem based on size (non-random)
- Solution: Randomly connect to a node (easy)
- + Randomly connect to the node's friends (also easy)
- Scale-free networks = food on the table for physicists

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