Complex Networks

Principles of Complex Systems
Course CSYS/MATH 300, Fall, 2009

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Outline

Overview of Complex Networks
Basic definitions
Examples of Complex Networks
Properties of Complex Networks
Nutshell
Basic models of complex networks
Generalized random networks
Scale-free networks
Small-world networks
Generalized affiliation networks
References

net•work |ˈnetˌwərk|
noun
1 an arrangement of intersecting horizontal and vertical lines.
- a complex system of roads, railroads, or other transportation routes : *a network of railroads.*

2 a group or system of interconnected people or things : *a trade network.*
- a group of people who exchange information, contacts, and experience for professional or social purposes : *a support network.*
- a group of broadcasting stations that connect for the simultaneous broadcast of a program : *the introduction of a second TV network* | [as adj.] *network television.*
- a number of interconnected computers, machines, or operations : *specialized computers that manage multiple outside connections to a network* | *a local cellular phone network.*
- a system of connected electrical conductors.

verb [trans.]
connect as or operate with a network : *the stock exchanges have proven to be resourceful in networking these deals.*
- link (machines, esp. computers) to operate interactively : [as adj.] (**networked**) *networked workstations.*
- [intrans.] [often as n.] (**networking**) interact with other people to exchange information and develop contacts, esp. to further one's career : *the skills of networking, bargaining, and negotiation.*

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Thesaurus deliciousness:

network
noun
1 *a network of arteries* WEB, lattice, net, matrix, mesh, crisscross, grid, reticulum, reticulation; Anatomy plexus.
2 *a network of lanes* MAZE, labyrinth, warren, tangle.
3 *a network of friends* SYSTEM, complex, nexus, web, webwork.

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Ancestry:

From Keith Briggs's excellent etymological investigation: (⊞)

- Opus reticulatum:
- A Latin origin?

[http://serialconsign.com/2007/11/we-put-net-network]

Ancestry:

First known use: Geneva Bible, 1560

'And thou shalt make unto it a grate like networke of brass (Exodus xxvii 4).'

From the OED via Briggs:

- 1658–: reticulate structures in animals
- 1839–: rivers and canals
- 1869–: railways
- 1883–: distribution network of electrical cables
- 1914–: wireless broadcasting networks

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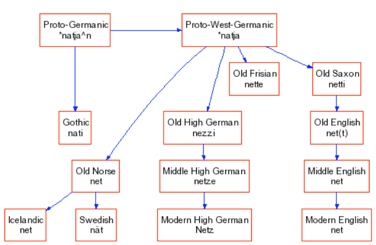
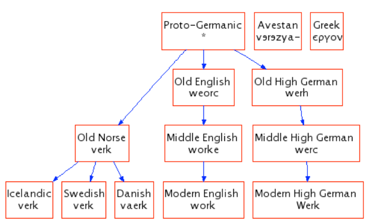
Ancestry:

Net and Work are venerable old words:

- 'Net' first used to mean spider web (King Ælfréd, 888).
- 'Work' appear to have long meant purposeful action.



The network of Germanic `net' words



The network of `work' words

- 'Network' = something built based on the idea of natural, flexible lattice or web.
- c.f., ironwork, stonework, fretwork.

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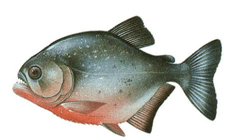
Key Observation:

- Many complex systems can be viewed as complex networks of physical or abstract interactions.
- Opens door to mathematical and numerical analysis.
- Dominant approach of last decade of a theoretical-physics/stat-mechish flavor.
- Mindboggling amount of work published on complex networks since 1998...
- ... largely due to your typical theoretical physicist:



- *Piranha physicus*
- Hunt in packs.
- Feast on new and interesting ideas (see chaos, cellular automata, ...)

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Popularity (according to ISI)

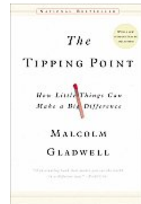
"Collective dynamics of 'small-world' networks" [28]

- Watts and Strogatz
  Nature, 1998
- ≈ 3752 citations (as of June 5, 2009)
- Over 1100 citations in 2008 alone.

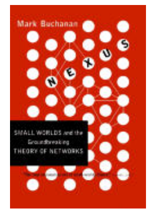
"Emergence of scaling in random networks" [3]

- Barabási and Albert
  Science, 1999
- ≈ 3860 citations (as of June 5, 2009)
- Over 1100 citations in 2008 alone.

Popularity according to books:



The Tipping Point: How Little Things can make a Big Difference—Malcolm Gladwell [12]



Nexus: Small Worlds and the Groundbreaking Science of Networks—Mark Buchanan

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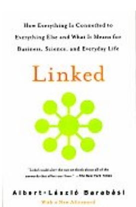
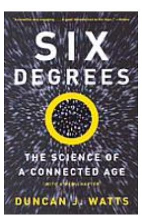


Linked: How Everything Is Connected to Everything Else and What It Means—Albert-Laszlo Barabási



Six Degrees: The Science of a Connected Age—Duncan Watts [26]

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Numerous others:

- Complex Social Networks—F. Vega-Redondo [25]
- Fractal River Basins: Chance and Self-Organization—I. Rodríguez-Iturbe and A. Rinaldo [20]
- Random Graph Dynamics—R. Durette
- Scale-Free Networks—Guido Caldarelli
- Evolution and Structure of the Internet: A Statistical Physics Approach—Romu Pastor-Satorras and Alessandro Vespignani
- Complex Graphs and Networks—Fan Chung
- Social Network Analysis—Stanley Wasserman and Kathleen Faust
- Handbook of Graphs and Networks—Eds: Stefan Bornholdt and H. G. Schuster [6]
- Evolution of Networks—S. N. Dorogovtsev and J. F. F. Mendes [11]

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More observations

- But surely networks aren't new...
- Graph theory is well established...
- Study of social networks started in the 1930's...
- So why all this 'new' research on networks?
- Answer: Oodles of Easily Accessible Data.
- We can now inform (alas) our theories with a much more measurable reality.*
- A worthy goal: establish mechanistic explanations.

** If this is upsetting, maybe string theory is for you...*

More observations

- Web-scale data sets can be overly exciting.

Witness:

- The End of Theory: The Data Deluge Makes the Scientific Theory Obsolete (Anderson, Wired) (⊞)
- "The Unreasonable Effectiveness of Data," Halevy et al.[13].

But:

- For scientists, description is only part of the battle.
- We still need to understand.

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Super Basic definitions

Nodes = A collection of entities which have properties that are somehow related to each other

- e.g., people, forks in rivers, proteins, webpages, organisms,...

Links = Connections between nodes

- Links may be directed or undirected.
- Links may be binary or weighted.

Other spiffing words: vertices and edges.

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Super Basic definitions

Node degree = Number of links per node

- Notation: Node i's degree = k_i.
- $k_i = 0,1,2,\ldots$.
- Notation: the average degree of a network = $\langle k \rangle$ (and sometimes z)
- Connection between number of edges m and average degree:
$$\langle k \rangle = \frac{2m}{N}.$$
- Defn: $\mathcal{N}_i$ = the set of i's k_i neighbors

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Super Basic definitions

Adjacency matrix:

- We represent a directed network by a matrix A with link weight a_{ij} for nodes i and j in entry (i,j).
- e.g.,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

- (n.b., for numerical work, we always use sparse matrices.)

Examples

So what passes for a complex network?

- Complex networks are large (in node number)
- Complex networks are sparse (low edge to node ratio)
- Complex networks are usually dynamic and evolving
- Complex networks can be social, economic, natural, informational, abstract, ...

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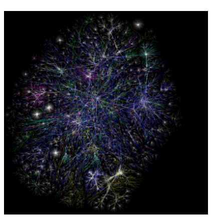
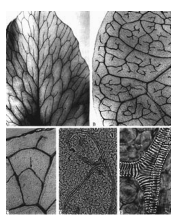
Examples

Physical networks

- River networks
- Neural networks
- Trees and leaves
- Blood networks
- The Internet
- Road networks
- Power grids





- Distribution (branching) versus redistribution (cyclical)

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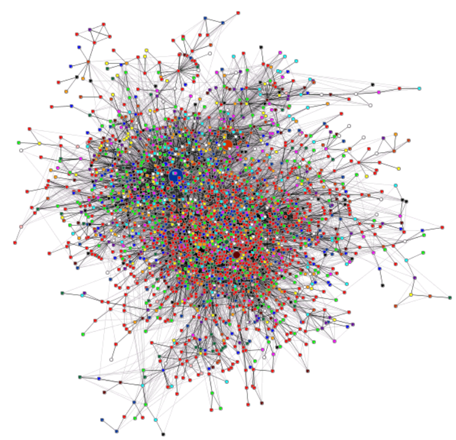
Examples

Interaction networks

- The Blogosphere
- Biochemical networks
- Gene-protein networks
- Food webs: who eats whom
- The World Wide Web (?)
- Airline networks
- Call networks (AT&T)
- The Media



datamining.typepad.com (⊞)

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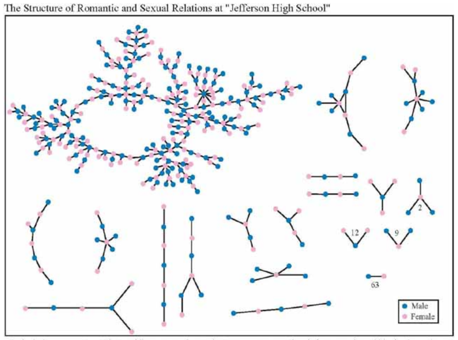
Examples

Interaction networks: social networks

- Snogging
- Friendships
- Acquaintances
- Boards and directors
- Organizations
- facebook (⊞) twitter (⊞),

The Structure of Romantic and Sexual Relations at "Jefferson High School"



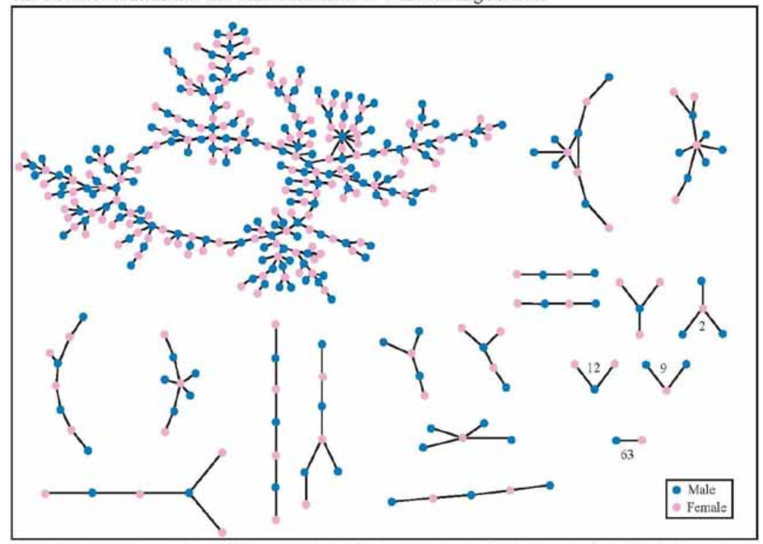
Each circle represents a student and lines connecting students represent romantic relations occuring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).

(Bearman *et al.*, 2004)

- 'Remotely sensed' by: email activity, instant messaging, phone logs (*cough*).

Examples

The Structure of Romantic and Sexual Relations at "Jefferson High School"



Each circle represents a student and lines connecting students represent romantic relations occuring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).

Examples

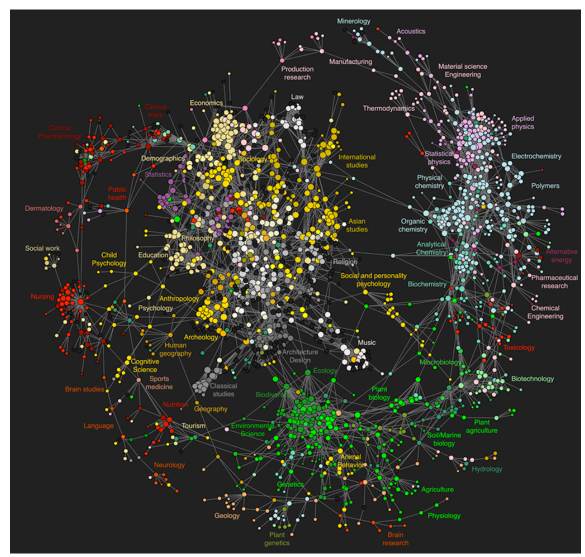
Relational networks

- Consumer purchases (Wal-Mart: $\approx$ 1 petabyte $= 10^{15}$ bytes)
- Thesauri: Networks of words generated by meanings
- Knowledge/Databases/Ideas
- Metadata—Tagging: del.icio.us (⊞) flickr (⊞)

common tags cloud | list

community daily dictionary education **encyclopedia**
english free imported info information internet knowledge
learning news **reference** research resource
resources search tools useful web web2.0 **wiki**
wikipedia

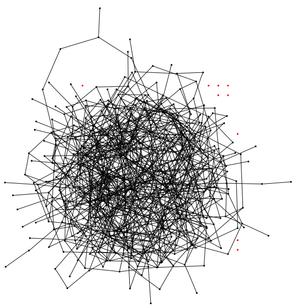
Clickworthy Science:



Bollen et al. [5]

A notable feature of large-scale networks:

- Graphical renderings are often just a big mess.



⇐ Typical hairball

- number of nodes $N = 500$
- number of edges $m = 1000$
- average degree $\langle k \rangle = 4$

- And even when renderings somehow look good: "That is a very graphic analogy which aids understanding wonderfully while being, strictly speaking, wrong in every possible way" said Ponder [Stibbons] —*Making Money*, T. Pratchett.
- We need to extract digestible, meaningful aspects.

Properties

Some key features of real complex networks:

- Degree distribution
- Assortativity
- Homophily
- Clustering
- Motifs
- Modularity
- Concurrency
- Hierarchical scaling
- Network distances
- Centrality
- Efficiency
- Robustness

- Coevolution of network structure and processes on networks.

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Properties

1. Degree distribution P_k

- P_k is the probability that a randomly selected node has degree k
- Big deal: Form of P_k key to network's behavior
- ex 1: Erdös-Rényi random networks have a Poisson distribution:

$$P_k = e^{-\langle k \rangle} \langle k \rangle^k / k!$$

- ex 2: "Scale-free" networks: $P_k \propto k^{-\gamma} \Rightarrow$ 'hubs'
- We'll come back to this business soon...

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Properties

2. Assortativity/3. Homophily:

- Social networks: Homophily (⊞) = birds of a feather
- e.g., degree is standard property for sorting: measure degree-degree correlations.
- Assortative network: [18] similar degree nodes connecting to each other.
  - *Often social: company directors, coauthors, actors.*
- Disassortative network: high degree nodes connecting to low degree nodes.
  - *Often techological or biological: Internet, protein interactions, neural networks, food webs.*

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Properties

4. Clustering:

- Your friends tend to know each other.
- Two measures:

$$C_1 = \left\langle \frac{\sum_{j_1 j_2 \in \mathcal{N}_i} a_{j_1 j_2}}{k_i(k_i-1)/2} \right\rangle_i \text{ due to Watts \& Strogatz [28]}$$

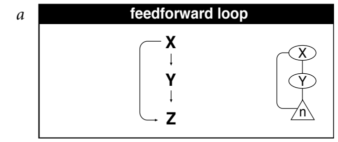
$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}} \text{ due to Newman [19]}$$

- C_1 is the average fraction of pairs of neighbors who are connected.
- Interpret C_2 as probability two of a node's friends know each other.

Properties

5. Motifs:

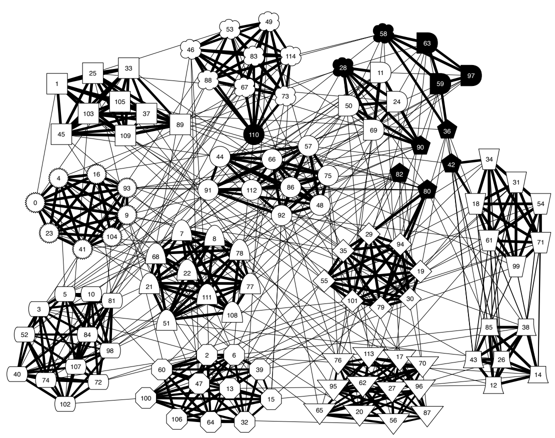
- Small, recurring functional subnetworks
- e.g., Feed Forward Loop:



Shen-Orr, Uri Alon, *et al.* [21]

Properties

6. modularity:



Clauset *et al.*, 2006 [8]: NCAA football

Properties

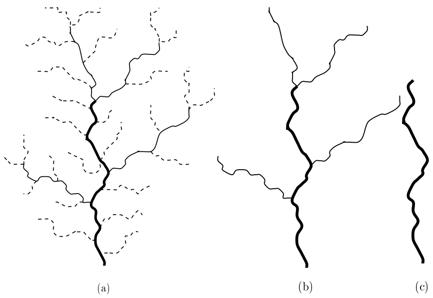
7. Concurrency:

- Transmission of a contagious element only occurs during contact [16]
- Rather obvious but easily missed in a simple model
- Dynamic property—static networks are not enough
- Knowledge of previous contacts crucial
- Beware cumulated network data!

Properties

8. Horton-Strahler stream ordering:

- Metrics for branching networks:
  - Method for ordering streams hierarchically
  - Reveals fractal nature of natural branching networks
  - Hierarchy is not pure but mixed (Tokunaga).[23, 10]
  - Major examples: rivers and blood networks.



- Beautifully described but poorly explained.

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Properties

9. Network distances:

(a) shortest path length d_{ij}:

- Fewest number of steps between nodes i and j.
- (Also called the chemical distance between i and j.)

(b) average path length $\langle d_{ij} \rangle$:

- Average shortest path length in whole network.
- Good algorithms exist for calculation.
- Weighted links can be accommodated.

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9. Network distances:

(c) Network diameter $d_{\max}$:

- Maximum shortest path length in network.

(d) Closeness $d_{\mathrm{cl}} = [\sum_{ij} d_{ij}^{-1} / \binom{n}{2}]^{-1}$:

- Average 'distance' between any two nodes.
- Closeness handles disconnected networks ($d_{ij} = \infty$)
- $d_{\mathrm{cl}} = \infty$ only when all nodes are isolated.

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Properties

10. Centrality:

- Many such measures of a node's 'importance.'
- ex 1: Degree centrality: k_i.
- ex 2: Node i's betweenness
  = fraction of shortest paths that pass through i.
- ex 3: Edge ℓ's betweenness
  = fraction of shortest paths that travel along ℓ.
- ex 4: Recursive centrality: Hubs and Authorities (Jon Kleinberg[15])

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Nutshell:

Overview Key Points:

- The field of complex networks came into existence in the late 1990s.
- Explosion of papers and interest since 1998/99.
- Hardened up much thinking about complex systems.
- Specific focus on networks that are large-scale, sparse, natural or man-made, evolving and dynamic, and (crucially) measurable.
- Three main (blurred) categories:
  1. Physical (e.g., river networks),
  2. Interactional (e.g., social networks),
  3. Abstract (e.g., thesauri).

Nutshell:

Overview Key Points (cont.):

- Obvious connections with the vast extant field of graph theory.
- But focus on dynamics is more of a physics/stat-mech/comp-sci flavor.
- Two main areas of focus:
  1. Description: Characterizing very large networks
  2. Explanation: Micro story ⇒ Macro features
- Some essential structural aspects are understood: degree distribution, clustering, assortativity, group structure, overall structure,...
- Still much work to be done, especially with respect to dynamics...

Models

Some important models:

1. generalized random networks
2. scale-free networks
3. small-world networks
4. statistical generative models (p^*)
5. generalized affiliation networks

Models

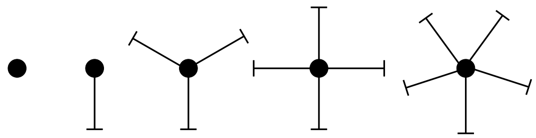
Generalized random networks:

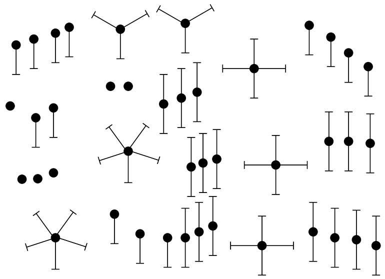
- Arbitrary degree distribution P_k.
- Create (unconnected) nodes with degrees sampled from P_k.
- Wire nodes together randomly.
- Create ensemble to test deviations from randomness.

Building random networks: Stubs

Phase 1:

- Idea: start with a soup of unconnected nodes with stubs (half-edges):



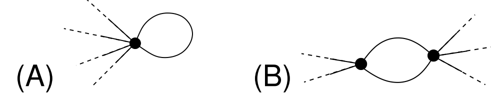


- Randomly select stubs (not nodes!) and connect them.
- Must have an even number of stubs.
- Initially allow self- and repeat connections.

Building random networks: First rewiring

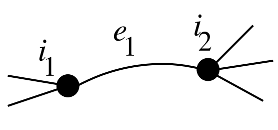
Phase 2:

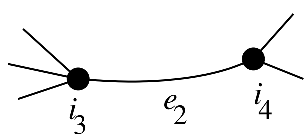
- Now find any (A) self-loops and (B) repeat edges and randomly rewire them.



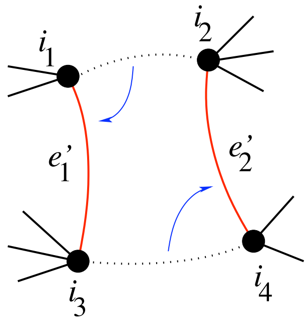
- Being careful: we can't change the degree of any node, so we can't simply move links around.
- Simplest solution: randomly rewire two edges at a time.

General random rewiring algorithm





- Randomly choose two edges. (Or choose problem edge and a random edge)
- Check to make sure edges are disjoint.



- Rewire one end of each edge.
- Node degrees do not change.
- Works if e_1 is a self-loop or repeated edge.
- Same as finding on/off/on/off 4-cycles. and rotating them.

Sampling random networks

Phase 2:

- Use rewiring algorithm to remove all self and repeat loops.

Phase 3:

- Randomize network wiring by applying rewiring algorithm liberally.
- Rule of thumb: # Rewirings $\simeq 10 \times$ # edges [17].

Scale-free networks

- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma} \text{ for 'large' } k$$

- One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999: "Emergence of scaling in random networks" [3]
- Somewhat misleading nomenclature...

Scale-free networks

- Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, . . . (non-physical)
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not. . .

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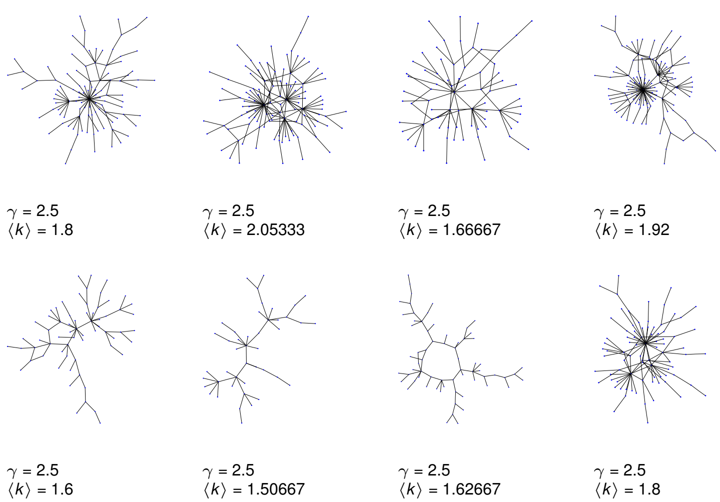
Random networks: largest components



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Scale-free networks

The big deal:

- We move beyond describing networks to finding mechanisms for why certain networks are the way they are.

A big deal for scale-free networks:

- How does the exponent γ depend on the mechanism?
- Do the mechanism details matter?

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BA model

- Barabási-Albert model = BA model.
- Key ingredients:
  Growth and Preferential Attachment (PA).
- Step 1: start with m_0 disconnected nodes.
- Step 2:
  1. Growth—a new node appears at each time step $t = 0, 1, 2, \ldots$.
  2. Each new node makes m links to nodes already present.
  3. Preferential attachment—Probability of connecting to ith node is $\propto k_i$.
- In essence, we have a rich-gets-richer scheme.

BA model

- Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- Definition: $P_{\mathrm{attach}}(k, t)$ is the attachment probability.
- For the original model:

$$P_{\mathrm{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=0}^{k_{\max}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

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Approximate analysis

- When $(N+1)$th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1} - k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- Approximate $k_{i,N+1} - k_{i,N}$ with $\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where $t = N(t) - m_0$.

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Approximate analysis

- Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

- The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t} k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m \frac{k_i(t)}{2mt} = \frac{1}{2t} k_i(t)$$

- Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i\, t^{1/2}.}$$

- Next find c_i ...

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Approximate analysis

- Know ith node appears at time

$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

- So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \quad \text{for } t \ge t_{i,\text{start}}.$$

- All node degrees grow as $t^{1/2}$ but later nodes have larger $t_{i,\text{start}}$ which flattens out growth curve.
- Early nodes do best (First-mover advantage).

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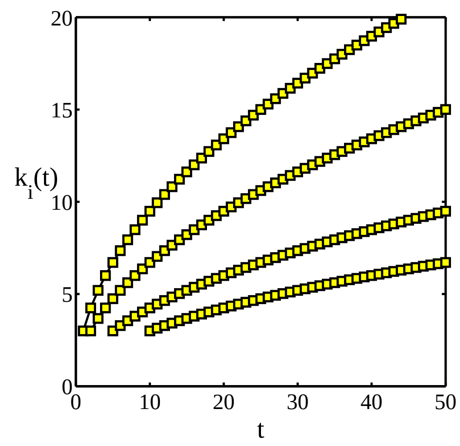
Approximate analysis



- $m = 3$
- $t_{i,\text{start}} = 1, 2, 5,$ and 10.

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Degree distribution

- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly:

$$\mathbf{Pr}(t_{i,\text{start}})\mathrm{d}t_{i,\text{start}} \simeq \frac{\mathrm{d}t_{i,\text{start}}}{t}$$

- Also use

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

Transform variables—Jacobian:

$$\frac{\mathrm{d}t_{i,\text{start}}}{\mathrm{d}k_i} = -2\frac{m^2 t}{k_i(t)^3}.$$

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Degree distribution

- $$\mathbf{Pr}(k_i)\mathrm{d}k_i = \mathbf{Pr}(t_{i,\text{start}})\mathrm{d}t_{i,\text{start}}$$
- $$= \mathbf{Pr}(t_{i,\text{start}})\mathrm{d}k_i \left|\frac{\mathrm{d}t_{i,\text{start}}}{\mathrm{d}k_i}\right|$$
- $$= \frac{1}{t}\mathrm{d}k_i \, 2\frac{m^2 t}{k_i(t)^3}$$
- $$= 2\frac{m^2}{k_i(t)^3}\mathrm{d}k_i$$
- $$\propto k_i^{-3}\mathrm{d}k_i\,.$$

Degree distribution

- We thus have a very specific prediction of $\mathbf{Pr}(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- $2 < \gamma < 3$: finite mean and 'infinite' variance (wild)
- In practice, $\gamma < 3$ means variance is governed by upper cutoff.
- $\gamma > 3$: finite mean and variance (mild)

Examples

WWW	$\gamma \simeq 2.1$ for in-degree
WWW	$\gamma \simeq 2.45$ for out-degree
Movie actors	$\gamma \simeq 2.3$
Words (synonyms)	$\gamma \simeq 2.8$

The Internets is a different business...

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Real data

From Barabási and Albert's original paper [3]:

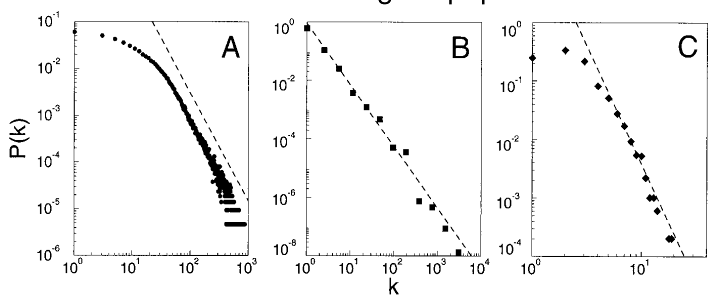

Fig. 1. The distribution function of connectivities for various large networks. **(A)** Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. **(B)** WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). **(C)** Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{actor} = 2.3$, (B) $\gamma_{www} = 2.1$ and (C) $\gamma_{power} = 4$.

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Things to do and questions

- Vary attachment kernel.
- Vary mechanisms:
  1. Add edge deletion
  2. Add node deletion
  3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter?
- The answer is (surprisingly) yes. More later re Zipf.

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Preferential attachment

- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- For example: If $P_{\mathrm{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- We need to know what everyone's degree is...
- PA is $\therefore$ an outrageous assumption of node capability.
- But a very simple mechanism saves the day…

Preferential attachment through randomness

- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- Assuming the existing network is random, we know probability of a random friend having degree k is

$$Q_k \propto kP_k$$

- So rich-gets-richer scheme can now be seen to work in a natural way.

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Robustness

- System robustness and system robustness.
- Albert et al., Nature, 2000:
  "Error and attack tolerance of complex networks" [2]

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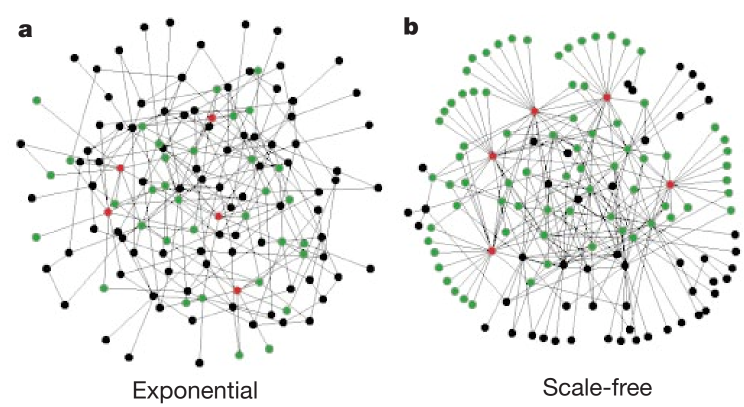
Robustness

- Standard random networks (Erdös-Rényi) versus Scale-free networks



from Albert et al., 2000

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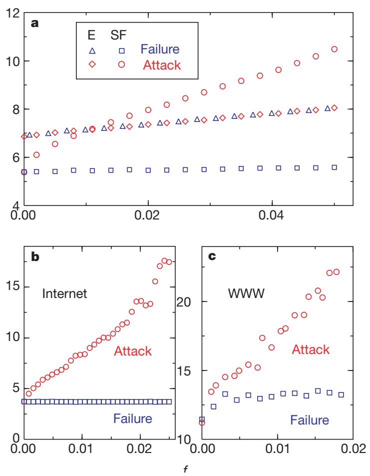
Robustness



from Albert et al., 2000

- Plots of network diameter as a function of fraction of nodes removed
- Erdös-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  1. Physically larger nodes that may be harder to 'target'
  2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

Some problems for people thinking about people?:

How are social networks structured?

- How do we define connections?
- How do we measure connections?
- (remote sensing, self-reporting)

What about the dynamics of social networks?

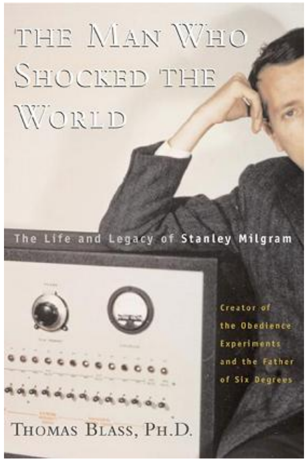
- How do social networks evolve?
- How do social movements begin?
- How does collective problem solving work?
- How is information transmitted through social networks?

Social Search

A small slice of the pie:

- Q. Can people pass messages between distant individuals using only their existing social connections?
- A. Apparently yes...

Milgram's social search experiment (1960s)



http://www.stanleymilgram.com

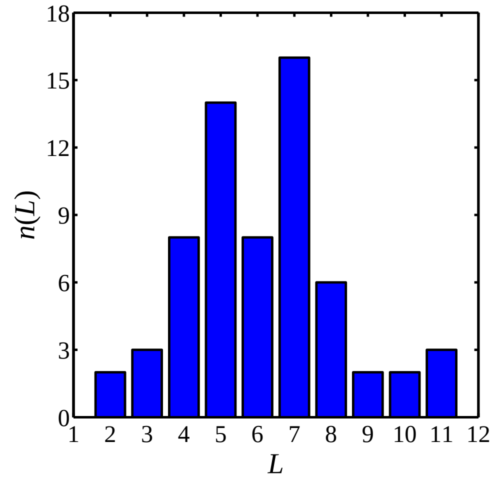
- Target person = Boston stockbroker.
- 296 senders from Boston and Omaha.
- 20% of senders reached target.
- chain length $\simeq$ 6.5.

Popular terms:

- The Small World Phenomenon;
- "Six Degrees of Separation."

The problem

Lengths of successful chains:



From Travers and Milgram (1969) in Sociometry:[24] "An Experimental Study of the Small World Problem."

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The problem

Two features characterize a social 'Small World':

1. Short paths exist

   and
2. People are good at finding them.

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Social Search

Milgram's small world experiment with e-mail [9]

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Social search—the Columbia experiment

- 60,000+ participants in 166 countries
- 18 targets in 13 countries including
  - a professor at an Ivy League university,
  - an archival inspector in Estonia,
  - a technology consultant in India,
  - a policeman in Australia, and
  - a veterinarian in the Norwegian army.
- 24,000+ chains

Social search—the Columbia experiment

- Milgram's participation rate was roughly 75%
- Email version: Approximately 37% participation rate.
- Probability of a chain of length 10 getting through:

$$.37^{10} \simeq 5 \times 10^{-5}$$

- $\Rightarrow$ 384 completed chains (1.6% of all chains).

Social search—the Columbia experiment

- Motivation/Incentives/Perception matter.
- If target *seems* reachable
  $\Rightarrow$ participation more likely.
- Small changes in attrition rates
  $\Rightarrow$ large changes in completion rates
- e.g., $\searrow$ 15% in attrition rate
  $\Rightarrow \nearrow$ 800% in completion rate

Social search—the Columbia experiment

Successful chains disproportionately used

- weak ties (Granovetter)
- professional ties (34% vs. 13%)
- ties originating at work/college
- target's work (65% vs. 40%)

. . . and disproportionately avoided

- hubs (8% vs. 1%) (+ no evidence of funnels)
- family/friendship ties (60% vs. 83%)

Geography $\rightarrow$ Work

Social search—the Columbia experiment

Senders of successful messages showed little absolute dependency on

- age, gender
- country of residence
- income
- religion
- relationship to recipient

Range of completion rates for subpopulations: 30% to 40%

Social search—the Columbia experiment

Nevertheless, some weak discrepencies do exist...

An above average connector:

Norwegian, secular male, aged 30-39, earning over $100K, with graduate level education working in mass media or science, who uses relatively weak ties to people they met in college or at work.

A below average connector:

Italian, Islamic or Christian female earning less than $2K, with elementary school education and retired, who uses strong ties to family members.

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Social search—the Columbia experiment

Mildly bad for continuing chain:

choosing recipients because "they have lots of friends" or because they will "likely continue the chain."

Why:

- Specificity important
- Successful links used relevant information. (e.g. connecting to someone who shares same profession as target.)

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Social search—the Columbia experiment

Basic results:

- $\langle L \rangle = 4.05$ for all completed chains
- L_* = Estimated 'true' median chain length (zero attrition)
- Intra-country chains: $L_* = 5$
- Inter-country chains: $L_* = 7$
- All chains: $L_* = 7$
- Milgram: $L_* \simeq 9$

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The social world appears to be small...

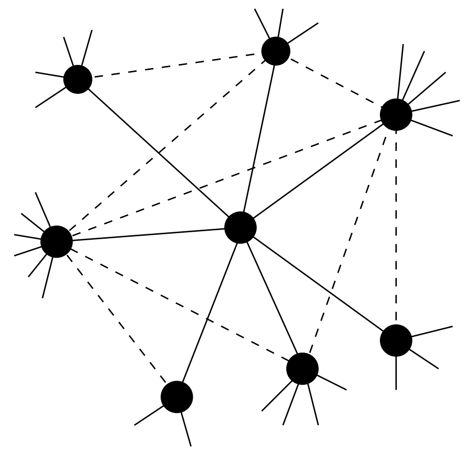
- Connected random networks have short average path lengths:

$$\langle d_{AB} \rangle \sim \log(N)$$

  N = population size,

  d_{AB} = distance between nodes A and B.
- But: social networks aren't random...

Simple socialness in a network:



Need "clustering" (your friends are likely to know each other):

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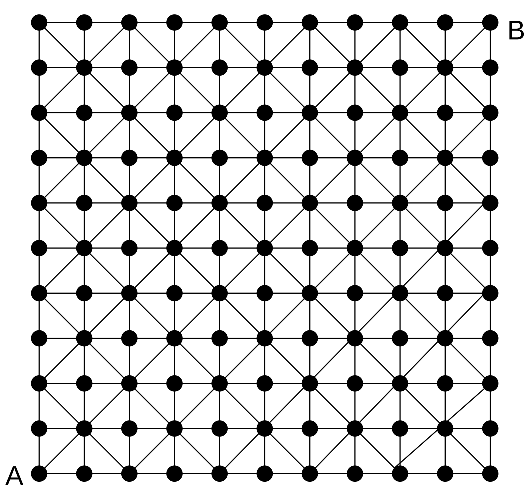
Non-randomness gives clustering:



$d_{AB} = 10 \rightarrow$ too many long paths.

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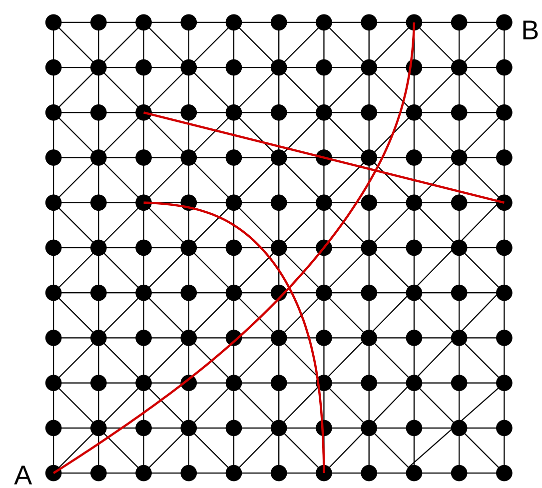
Randomness + regularity



Now have $d_{AB} = 3$

$\langle d \rangle$ decreases overall

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Small-world networks

Introduced by Watts and Strogatz (Nature, 1998) [28]
"Collective dynamics of 'small-world' networks."

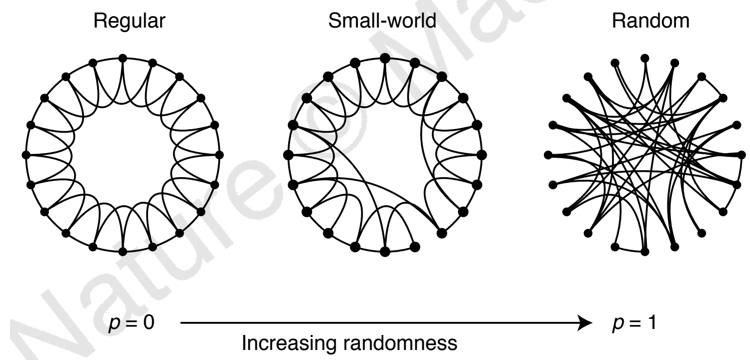
Small-world networks were found everywhere:

- neural network of C. elegans,
- semantic networks of languages,
- actor collaboration graph,
- food webs,
- social networks of comic book characters,...

Very weak requirements:

- local regularity + random short cuts

Toy model:



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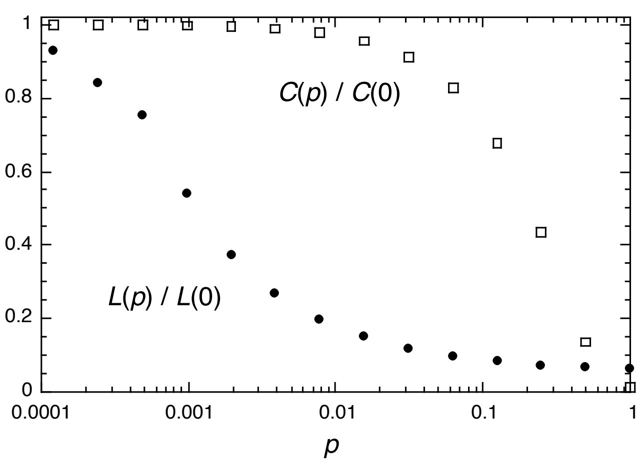
The structural small-world property:



- $L(p)$ = average shortest path length as a function of p
- $C(p)$ = average clustring as a function of p

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Previous work—finding short paths

But are these short cuts findable?

Nope.

Nodes cannot find each other quickly
with any local search method.

Need a more sophisticated model...

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Previous work—finding short paths

- What can a local search method reasonably use?
- How to find things without a map?
- Need some measure of distance between friends and the target.

Some possible knowledge:

- Target's identity
- Friends' popularity
- Friends' identities
- Where message has been

Previous work—finding short paths

Jon Kleinberg (Nature, 2000) [14]
"Navigation in a small world."

Allowed to vary:

1. local search algorithm
   and
2. network structure.

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Previous work—finding short paths

Kleinberg's Network:

1. Start with regular d-dimensional cubic lattice.
2. Add local links so nodes know all nodes within a distance q.
3. Add m short cuts per node.
4. Connect i to j with probability

$$p_{ij} \propto x_{ij}^{-\alpha}.$$

- $\alpha = 0$: random connections.
- α large: reinforce local connections.
- $\alpha = d$: same number of connections at all scales.

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Previous work—finding short paths

Theoretical optimal search:

- "Greedy" algorithm.
- Same number of connections at all scales: $\alpha = d$.

Search time grows slowly with system size (like $\log^2 N$).

But: social networks aren't lattices plus links.

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Previous work—finding short paths

- If networks have hubs can also search well: Adamic et al. (2001) [1]

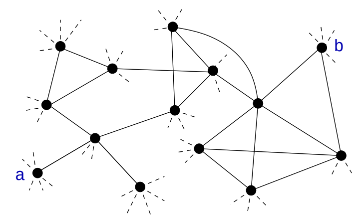
$$P(k_i) \propto k_i^{-\gamma}$$

where k = degree of node i (number of friends).

- Basic idea: get to hubs first (airline networks).
- But: hubs in social networks are limited.

The problem

If there are no hubs and no underlying lattice, how can search be efficient?



Which friend of a is closest to the target b?

What does 'closest' mean?

What is 'social distance'?

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Models

One approach: incorporate identity.

Identity is formed from attributes such as:

- Geographic location
- Type of employment
- Religious beliefs
- Recreational activities.

Groups are formed by people with at least one similar attribute.

Attributes ⇔ Contexts ⇔ Interactions ⇔ Networks.

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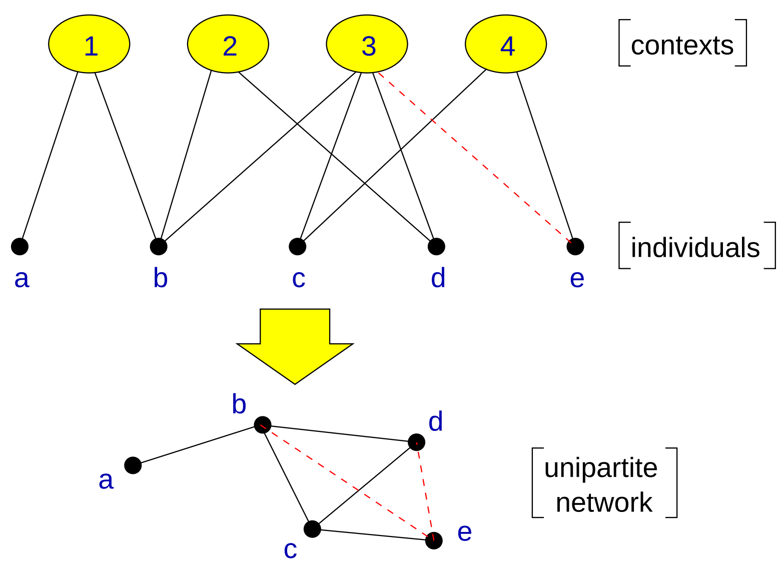
Social distance—Bipartite affiliation networks



Bipartite affiliation networks: boards and directors, movies and actors.

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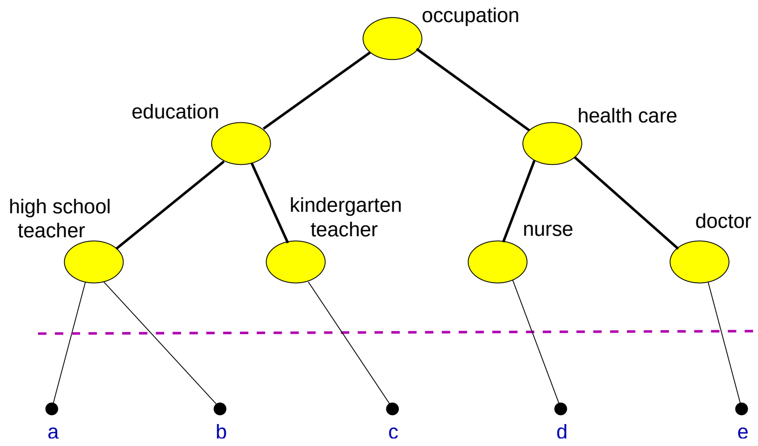
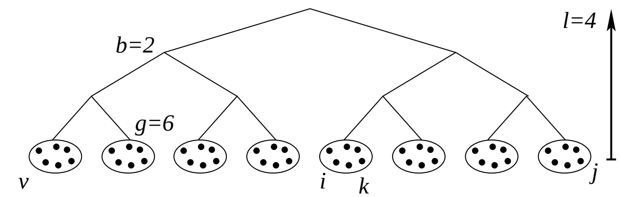
Social distance—Context distance



Models

Distance between two individuals x_{ij} is the height of lowest common ancestor.



$x_{ij} = 3$, $x_{ik} = 1$, $x_{iv} = 4$.

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Models

- Individuals are more likely to know each other the closer they are within a hierarchy.
- Construct z connections for each node using

$$p_{ij} = c \exp\{-\alpha x_{ij}\}.$$

- $\alpha = 0$: random connections.
- α large: local connections.

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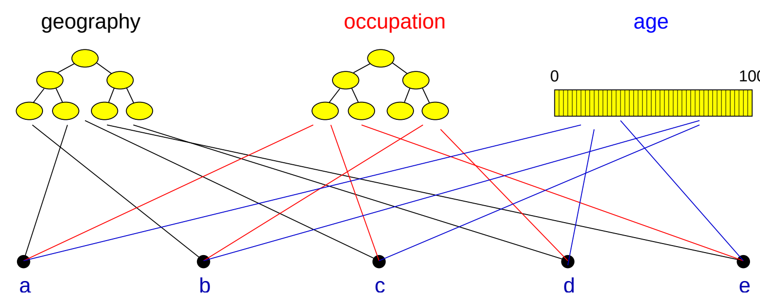
Models

Generalized affiliation networks



- Blau & Schwartz [4], Simmel [22], Breiger [7], Watts *et al.* [27]

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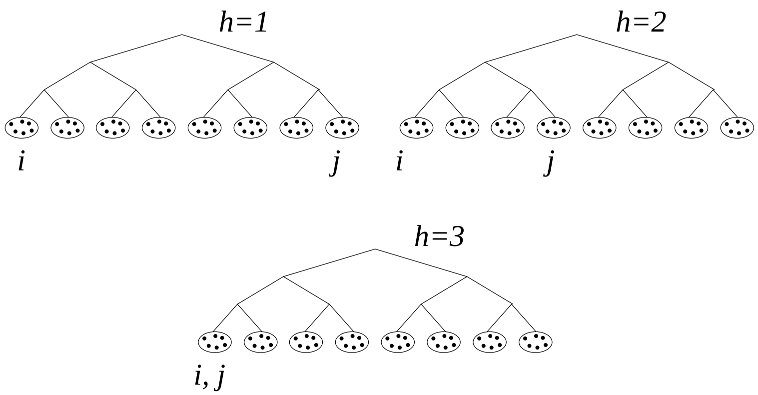
The model



$\vec{v}_i = [1\ 1\ 1]^T$, $\vec{v}_j = [8\ 4\ 1]^T$

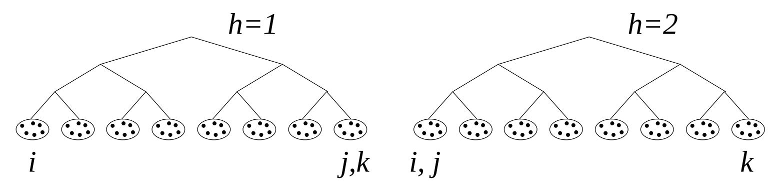
$x_{ij}^1 = 4$, $x_{ij}^2 = 3$, $x_{ij}^3 = 1$.

Social distance:

$$y_{ij} = \min_h x_{ij}^h.$$

The model

Triangle inequality doesn't hold:



$y_{ik} = 4 > y_{ij} + y_{jk} = 1 + 1 = 2.$

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The model

- Individuals know the identity vectors of
  1. themselves,
  2. their friends,
     and
  3. the target.
- Individuals can estimate the social distance between their friends and the target.
- Use a greedy algorithm + allow searches to fail randomly.

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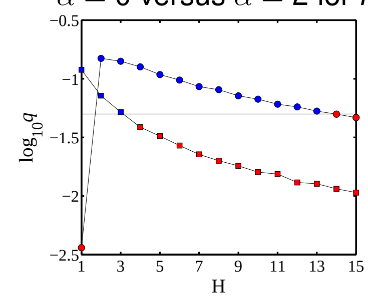
The model-results—searchable networks

$\alpha = 0$ versus $\alpha = 2$ for $N \simeq 10^5$:



$q \geq r$

$q < r$

$r = 0.05$

q = probability an arbitrary message chain reaches a target.

- A few dimensions help.
- Searchability decreases as population increases.
- Precise form of hierarchy largely doesn't matter.

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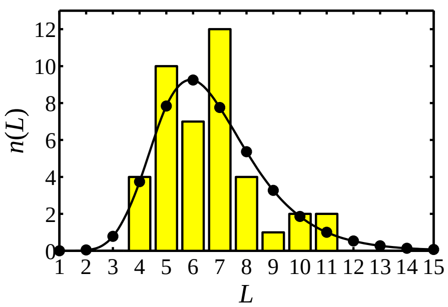
The model-results

Milgram's Nebraska-Boston data:



Model parameters:

- $N = 10^8$,
- $z = 300$, $g = 100$,
- $b = 10$,
- $\alpha = 1$, $H = 2$;

- $\langle L_{\text{model}} \rangle \simeq 6.7$
- $L_{\text{data}} \simeq 6.5$

Social search—Data

Adamic and Adar (2003)

- For HP Labs, found probability of connection as function of organization distance well fit by exponential distribution.
- Probability of connection as function of real distance $\propto 1/r$.

Social Search—Real world uses

- Tags create identities for objects
- Website tagging: `http://www.del.icio.us`
- (e.g., Wikipedia)
- Photo tagging: `http://www.flickr.com`
- Dynamic creation of metadata plus links between information objects.
- Folksonomy: collaborative creation of metadata

Social Search—Real world uses

Recommender systems:

- Amazon uses people's actions to build effective connections between books.
- Conflict between 'expert judgments' and tagging of the hoi polloi.

Nutshell

- Bare networks are typically unsearchable.
- Paths are findable if nodes understand how network is formed.
- Importance of identity (interaction contexts).
- Improved social network models.
- Construction of peer-to-peer networks.
- Construction of searchable information databases.

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