

Lognormals and friends

Principles of Complex Systems

Course CSYS/MATH 300, Fall, 2009

Prof. Peter Dodds

Dept. of Mathematics & Statistics
Center for Complex Systems :: Vermont Advanced Computing Center
University of Vermont



Lognormals

- Empirical Confusability
- Random Multiplicative Growth Model
- Random Growth with Variable Lifespan

References

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

There are other heavy-tailed distributions:

1. Lognormal
2. Stretched exponential (Weibull)
3. ... (Gamma)

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- ▶ $\ln x$ is distributed according to a normal distribution with mean μ and variance σ .
- ▶ Appears in economics and biology where growth increments are distributed normally.

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu + \frac{1}{2}\sigma^2}, \quad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}, \quad \text{mode}_{\text{lognormal}} = e^{\mu - \sigma^2}.$$

All moments of lognormals are finite.

Lognormals

- Empirical Confusability
- Random Multiplicative Growth Model
- Random Growth with Variable Lifespan

References

Derivation from a normal distribution

Take Y as distributed normally:

$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma} dy \exp\left(-\frac{(y - \mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

▶ Transform according to $P(x)dx = P(y)dy$:

$$\frac{dy}{dx} = 1/x \Rightarrow dy = dx/x$$

$$\Rightarrow P(x)dx = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

Lognormals

Empirical Contagability

Random Multiplicative

Growth Model

Random Growth with

Variable Lifespan

References

Derivation from a normal distribution

Take Y as distributed normally:



$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma} dy \exp\left(-\frac{(y - \mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

▶ Transform according to $P(x)dx = P(y)dy$:



$$\frac{dy}{dx} = 1/x \Rightarrow dy = dx/x$$



$$\Rightarrow P(x)dx = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

Lognormals

Empirical Confusability

Random Multiplicative

Growth Model

Random Growth with

Variable Lifespan

References

Derivation from a normal distribution

Take Y as distributed normally:



$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma} dy \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

▶ Transform according to $P(x)dx = P(y)dy$:



$$\frac{dy}{dx} = 1/x \Rightarrow dy = dx/x$$



$$\Rightarrow P(x)dx = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

Lognormals

Empirical Contagability

Random Multiplicative

Growth Model

Random Growth with

Variable Lifespan

References

Derivation from a normal distribution

Take Y as distributed normally:



$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma} dy \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

▶ Transform according to $P(x)dx = P(y)dy$:



$$\frac{dy}{dx} = 1/x \Rightarrow dy = dx/x$$



$$\Rightarrow P(x)dx = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

Lognormals

Empirical Confusability

Random Multiplicative

Growth Model

Random Growth with

Variable Lifespan

References

Derivation from a normal distribution

Take Y as distributed normally:



$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma} dy \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

▶ Transform according to $P(x)dx = P(y)dy$:



$$\frac{dy}{dx} = 1/x \Rightarrow dy = dx/x$$



$$\Rightarrow P(x)dx = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

Lognormals

Empirical Confusability

Random Multiplicative

Growth Model

Random Growth with

Variable Lifespan

References

Derivation from a normal distribution

Take Y as distributed normally:



$$P(y)dy = \frac{1}{\sqrt{2\pi}\sigma} dy \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

▶ Transform according to $P(x)dx = P(y)dy$:



$$\frac{dy}{dx} = 1/x \Rightarrow dy = dx/x$$



$$\Rightarrow P(x)dx = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

Lognormals

Empirical Confusability

Random Multiplicative

Growth Model

Random Growth with

Variable Lifespan

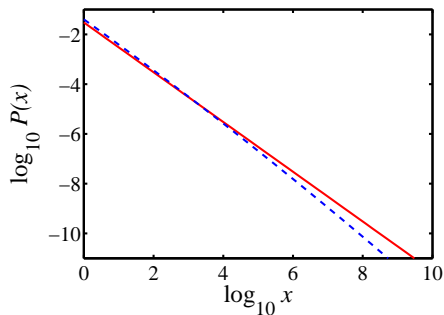
References

Confusion between lognormals and pure power laws

Lognormals

Empirical Confusability
Random Multiplicative Growth Model
Random Growth with Variable Lifespan

References



Near agreement over four orders of magnitude!

- ▶ For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- ▶ For power law (red), $\gamma = 1$ and $c = 0.03$.

What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp \left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2} (\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$

▶ \Rightarrow If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp \left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2} (\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$

▶ \Rightarrow If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp \left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$

▶ \Rightarrow If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp \left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2} (\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$

▶ \Rightarrow If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

What's happening:

$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp \left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}$$

$$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

$$= -\frac{1}{2\sigma^2} (\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$

▶ \Rightarrow If $\sigma^2 \gg 1$ and μ ,

$$\ln P(x) \sim -\ln x + \text{const.}$$

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

What's happening:



$$\ln P(x) = \ln \left\{ \frac{1}{x\sqrt{2\pi}\sigma} \exp \left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right) \right\}$$



$$= -\ln x - \ln \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$



$$= -\frac{1}{2\sigma^2} (\ln x)^2 + \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x - \ln \sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$

▶ \Rightarrow If $\sigma^2 \gg 1$ and μ ,



$$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

- ▶ Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$.

- ▶ This happens when (roughly)



$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x$$



$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$



$$\simeq 0.05(\sigma^2 - \mu)$$

- ▶ \Rightarrow If you find a -1 exponent,
you may have a lognormal distribution...

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

- ▶ Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$.
- ▶ This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$

$$\simeq 0.05(\sigma^2 - \mu)$$

- ▶ \Rightarrow If you find a -1 exponent, you may have a lognormal distribution...

Lognormals

Empirical Confusability
Random Multiplicative Growth Model
Random Growth with Variable Lifespan

References

- ▶ Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$.
- ▶ This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$

$$\simeq 0.05(\sigma^2 - \mu)$$

- ▶ \Rightarrow If you find a -1 exponent,
you may have a lognormal distribution...

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

- ▶ Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$.
- ▶ This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$

$$\simeq 0.05(\sigma^2 - \mu)$$

- ▶ \Rightarrow If you find a -1 exponent,
you may have a lognormal distribution...

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

- ▶ Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$.
- ▶ This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$

$$\simeq 0.05(\sigma^2 - \mu)$$

- ▶ \Rightarrow If you find a -1 exponent,
you may have a lognormal distribution...

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

- ▶ Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$.
- ▶ This happens when (roughly)

$$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1 \right) \ln x$$

$$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$

$$\simeq 0.05(\sigma^2 - \mu)$$

- ▶ \Rightarrow If you find a -1 exponent, you may have a lognormal distribution...

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

Generating lognormals:

Random multiplicative growth:



$$x_{n+1} = r x_n$$

where $r > 0$ is a random growth variable

- ▶ (Shrinkage is allowed)
- ▶ In log space, growth is by addition:

$$\ln x_{n+1} = \ln r + \ln x_n$$

- ▶ $\Rightarrow \ln x_n$ is normally distributed
- ▶ $\Rightarrow x_n$ is lognormally distributed

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

Generating lognormals:

Random multiplicative growth:



$$x_{n+1} = r x_n$$

where $r > 0$ is a random growth variable

- ▶ (Shrinkage is allowed)
- ▶ In log space, growth is by addition:

$$\ln x_{n+1} = \ln r + \ln x_n$$

- ▶ $\Rightarrow \ln x_n$ is normally distributed
- ▶ $\Rightarrow x_n$ is lognormally distributed

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

Generating lognormals:

Random multiplicative growth:



$$x_{n+1} = r x_n$$

where $r > 0$ is a random growth variable

- ▶ (Shrinkage is allowed)
- ▶ In log space, growth is by addition:

$$\ln x_{n+1} = \ln r + \ln x_n$$

- ▶ $\Rightarrow \ln x_n$ is normally distributed
- ▶ $\Rightarrow x_n$ is lognormally distributed

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

Generating lognormals:

Random multiplicative growth:



$$x_{n+1} = r x_n$$

where $r > 0$ is a random growth variable

- ▶ (Shrinkage is allowed)
- ▶ In log space, growth is by addition:

$$\ln x_{n+1} = \ln r + \ln x_n$$

- ▶ $\Rightarrow \ln x_n$ is normally distributed
- ▶ $\Rightarrow x_n$ is lognormally distributed

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

Generating lognormals:

Random multiplicative growth:



$$X_{n+1} = rX_n$$

where $r > 0$ is a random growth variable

- ▶ (Shrinkage is allowed)
- ▶ In log space, growth is by addition:

$$\ln x_{n+1} = \ln r + \ln x_n$$

- ▶ $\Rightarrow \ln x_n$ is normally distributed
- ▶ $\Rightarrow x_n$ is lognormally distributed

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Lognormals or power laws?

- ▶ Gibrat^[2] (1931) uses this argument to explain lognormal distribution of firm sizes
- ▶ Robert Axtell (2001) shows a power law fits the data very well^[1]

Lognormals

Empirical Contagability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Lognormals or power laws?

- ▶ Gibrat^[2] (1931) uses this argument to explain lognormal distribution of firm sizes
- ▶ Robert Axtell (2001) shows a power law fits the data very well^[1]

Lognormals

Empirical Contagability

Random Multiplicative
Growth Model

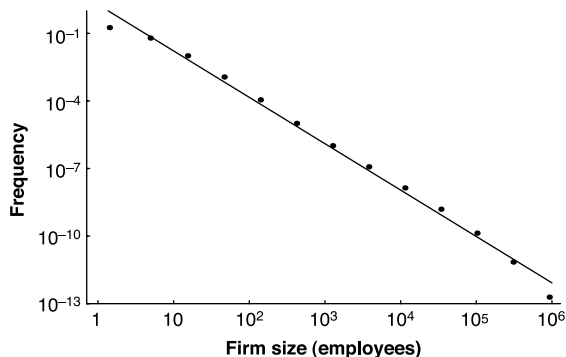
Random Growth with
Variable Lifespan

References

Lognormals or power laws?

- ▶ Gibrat^[2] (1931) uses this argument to explain lognormal distribution of firm sizes
- ▶ Robert Axtell (2001) shows a power law fits the data very well^[1]

$$\gamma \simeq 2$$



Lognormals

Empirical Contagability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

- ▶ Axtel (mis)cites Malcai et al.'s (1999) argument^[6] for why power laws appear with exponent $\gamma \simeq 1$
- ▶ The set up: N entities with size $x_i(t)$
- ▶ Generally:

$$x_i(t+1) = rx_i(t)$$

where r is drawn from some happy distribution

- ▶ Same as for lognormal but one extra piece:
- ▶ Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(rx_i(t), c \langle x_i \rangle)$$

- ▶ Axtel (mis)cites Malcai et al.'s (1999) argument^[6] for why power laws appear with exponent $\gamma \simeq 1$
- ▶ The set up: N entities with size $x_i(t)$
- ▶ Generally:

$$x_i(t+1) = rx_i(t)$$

where r is drawn from some happy distribution

- ▶ Same as for lognormal but one extra piece:
- ▶ Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(rx_i(t), c \langle x_i \rangle)$$

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

- ▶ Axtel (mis)cites Malcai et al.'s (1999) argument^[6] for why power laws appear with exponent $\gamma \simeq 1$
- ▶ The set up: N entities with size $x_i(t)$
- ▶ Generally:

$$x_i(t+1) = rx_i(t)$$

where r is drawn from some happy distribution

- ▶ Same as for lognormal but one extra piece:
- ▶ Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(rx_i(t), c \langle x_i \rangle)$$

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

- ▶ Axtel (mis)cites Malcai et al.'s (1999) argument^[6] for why power laws appear with exponent $\gamma \simeq 1$
- ▶ The set up: N entities with size $x_i(t)$
- ▶ Generally:

$$x_i(t+1) = rx_i(t)$$

where r is drawn from some happy distribution

- ▶ Same as for lognormal but one extra piece:
- ▶ Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(rx_i(t), c \langle x_i \rangle)$$

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

- ▶ Axtel (mis)cites Malcai et al.'s (1999) argument^[6] for why power laws appear with exponent $\gamma \simeq 1$
- ▶ The set up: N entities with size $x_i(t)$
- ▶ Generally:

$$x_i(t+1) = rx_i(t)$$

where r is drawn from some happy distribution

- ▶ Same as for lognormal but one extra piece:
- ▶ Each x_i cannot drop too low with respect to the other sizes:

$$x_i(t+1) = \max(rx_i(t), c \langle x_i \rangle)$$

An explanation

Some math later...

Insert question 1, assignment 5 (田)



$$\text{Find } P(x) \sim x^{-\gamma}$$



$$\text{where } N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$



$$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$$



$$\text{Which gives } \gamma \sim 1 + \frac{1}{1 - c}$$

▶ Groovy... c small $\Rightarrow \gamma \simeq 2$

Lognormals

Empirical Contagability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

An explanation

Some math later...

Insert question 1, assignment 5 (田)



$$\text{Find } P(x) \sim x^{-\gamma}$$



$$\text{where } N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$



$$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$$



$$\text{Which gives } \gamma \sim 1 + \frac{1}{1 - c}$$

▶ Groovy... c small $\Rightarrow \gamma \simeq 2$

Lognormals

Empirical Contagability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

An explanation

Some math later...

Insert question 1, assignment 5 (田)



$$\text{Find } P(x) \sim x^{-\gamma}$$



$$\text{where } N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$



$$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$$



$$\text{Which gives } \gamma \sim 1 + \frac{1}{1 - c}$$

▶ Groovy... c small $\Rightarrow \gamma \simeq 2$

Lognormals

Empirical Contagability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

An explanation

Some math later...

Insert question 1, assignment 5 (田)



$$\text{Find } P(x) \sim x^{-\gamma}$$



$$\text{where } N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$



$$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$$



$$\text{Which gives } \gamma \sim 1 + \frac{1}{1 - c}$$

▶ Groovy... c small $\Rightarrow \gamma \simeq 2$

Lognormals

Empirical Contagability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

An explanation

Some math later...

Insert question 1, assignment 5 (田)



$$\text{Find } P(x) \sim x^{-\gamma}$$



$$\text{where } N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$



$$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$$



$$\text{Which gives } \gamma \sim 1 + \frac{1}{1 - c}$$

▶ Groovy... c small $\Rightarrow \gamma \simeq 2$

Lognormals

Empirical Contagability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

An explanation

Some math later...

Insert question 1, assignment 5 (田)



$$\text{Find } P(x) \sim x^{-\gamma}$$



$$\text{where } N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$



$$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma - 2)}{(\gamma - 1)} \left[\frac{-1}{-(c/N)} \right]$$



$$\text{Which gives } \gamma \sim 1 + \frac{1}{1 - c}$$

▶ **Groovy...** c small $\Rightarrow \gamma \simeq 2$

Lognormals

Empirical Contagability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

Lognormals

Empirical Confusability

Random Multiplicative
Growth Model

Random Growth with
Variable Lifespan

References

References

Ages of firms/people/... may not be the same

- ▶ Allow the number of updates for each size x_i to vary
- ▶ Example: $P(t)dt = ae^{-at}dt$ where $t = \text{age}$.
- ▶ Back to no bottom limit: each x_i follows a lognormal
- ▶ Sizes are distributed as^[7]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- ▶ Now averaging different lognormal distributions.

The second tweak

Ages of firms/people/... may not be the same

- ▶ Allow the number of updates for each size x_i to vary
- ▶ Example: $P(t)dt = ae^{-at}dt$ where $t = \text{age}$.
- ▶ Back to no bottom limit: each x_i follows a lognormal
- ▶ Sizes are distributed as^[7]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- ▶ Now averaging different lognormal distributions.

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Ages of firms/people/... may not be the same

- ▶ Allow the number of updates for each size x_i to vary
- ▶ Example: $P(t)dt = ae^{-at}dt$ where $t = \text{age}$.
- ▶ Back to no bottom limit: each x_i follows a lognormal
- ▶ Sizes are distributed as^[7]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- ▶ Now averaging different lognormal distributions.

Ages of firms/people/... may not be the same

- ▶ Allow the number of updates for each size x_i to vary
- ▶ Example: $P(t)dt = ae^{-at}dt$ where $t = \text{age}$.
- ▶ Back to no bottom limit: each x_i follows a lognormal
- ▶ Sizes are distributed as^[7]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- ▶ Now averaging different lognormal distributions.

Ages of firms/people/... may not be the same

- ▶ Allow the number of updates for each size x_i to vary
- ▶ Example: $P(t)dt = ae^{-at}dt$ where $t = \text{age}$.
- ▶ Back to no bottom limit: each x_i follows a lognormal
- ▶ Sizes are distributed as^[7]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- ▶ Now averaging different lognormal distributions.

Ages of firms/people/... may not be the same

- ▶ Allow the number of updates for each size x_i to vary
- ▶ Example: $P(t)dt = ae^{-at}dt$ where $t = \text{age}$.
- ▶ Back to no bottom limit: each x_i follows a lognormal
- ▶ Sizes are distributed as^[7]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) dt$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- ▶ Now averaging different lognormal distributions.

Lognormals

Empirical Confusability
Random Multiplicative Growth Model
Random Growth with Variable Lifespan

References



$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x \sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) dt$$

- ▶ Insert question 2, assignment 5 (田)
- ▶ Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

Lognormals

Empirical Confusability
Random Multiplicative Growth Model
Random Growth with Variable Lifespan

References



$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x \sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) dt$$

▶ Insert question 2, assignment 5 (田)

▶ Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References



$$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x \sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) dt$$

- ▶ Insert question 2, assignment 5 (田)
- ▶ Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

The second tweak



$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

- ▶ Depends on sign of $\ln x/m$, i.e., whether $x/m > 1$ or $x/m < 1$.



$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } x/m > 1 \end{cases}$$

- ▶ 'Break' in scaling (not uncommon)
- ▶ Double-Pareto distribution (田)
- ▶ First noticed by Montroll and Shlesinger [8, 9]
- ▶ Later: Huberman and Adamic [4, 5]: Number of pages per website

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References



$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

- ▶ Depends on sign of $\ln x/m$, i.e., whether $x/m > 1$ or $x/m < 1$.



$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } x/m > 1 \end{cases}$$

- ▶ 'Break' in scaling (not uncommon)
- ▶ Double-Pareto distribution (⊕)
- ▶ First noticed by Montroll and Shlesinger [8, 9]
- ▶ Later: Huberman and Adamic [4, 5]: Number of pages per website

The second tweak



$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

- ▶ Depends on sign of $\ln x/m$, i.e., whether $x/m > 1$ or $x/m < 1$.



$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } x/m > 1 \end{cases}$$

- ▶ 'Break' in scaling (not uncommon)
- ▶ Double-Pareto distribution (⊕)
- ▶ First noticed by Montroll and Shlesinger [8, 9]
- ▶ Later: Huberman and Adamic [4, 5]: Number of pages per website

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References



$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

- ▶ Depends on sign of $\ln x/m$, i.e., whether $x/m > 1$ or $x/m < 1$.



$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } x/m > 1 \end{cases}$$

- ▶ **'Break' in scaling** (not uncommon)
- ▶ Double-Pareto distribution (⊕)
- ▶ First noticed by Montroll and Shlesinger [8, 9]
- ▶ Later: Huberman and Adamic [4, 5]: Number of pages per website



$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

- ▶ Depends on sign of $\ln x/m$, i.e., whether $x/m > 1$ or $x/m < 1$.



$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } x/m > 1 \end{cases}$$

- ▶ **'Break' in scaling** (not uncommon)
- ▶ Double-Pareto distribution (田)
- ▶ First noticed by Montroll and Shlesinger [8, 9]
- ▶ Later: Huberman and Adamic [4, 5]: Number of pages per website



$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

- ▶ Depends on sign of $\ln x/m$, i.e., whether $x/m > 1$ or $x/m < 1$.



$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } x/m > 1 \end{cases}$$

- ▶ **'Break' in scaling** (not uncommon)
- ▶ Double-Pareto distribution (田)
- ▶ First noticed by Montroll and Shlesinger [8, 9]
- ▶ Later: Huberman and Adamic [4, 5]: Number of pages per website



$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda}(\ln x/m)^2}$$

- ▶ Depends on sign of $\ln x/m$, i.e., whether $x/m > 1$ or $x/m < 1$.



$$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } x/m > 1 \end{cases}$$

- ▶ **'Break' in scaling** (not uncommon)
- ▶ Double-Pareto distribution (田)
- ▶ First noticed by Montroll and Shlesinger [8, 9]
- ▶ Later: Huberman and Adamic [4, 5]: Number of pages per website

Quick summary of these exciting developments

- ▶ Lognormals and power laws can be awfully similar
- ▶ Random Multiplicative Growth leads to lognormal distributions
- ▶ Enforcing a minimum size leads to a power law tail
- ▶ With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- ▶ Take home message: Be careful out there...

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Quick summary of these exciting developments

- ▶ Lognormals and power laws can be awfully similar
- ▶ Random Multiplicative Growth leads to lognormal distributions
- ▶ Enforcing a minimum size leads to a power law tail
- ▶ With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- ▶ Take home message: Be careful out there...

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Quick summary of these exciting developments

- ▶ Lognormals and power laws can be awfully similar
- ▶ Random Multiplicative Growth leads to lognormal distributions
- ▶ Enforcing a minimum size leads to a power law tail
- ▶ With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- ▶ Take home message: Be careful out there...

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Quick summary of these exciting developments

- ▶ Lognormals and power laws can be awfully similar
- ▶ Random Multiplicative Growth leads to lognormal distributions
- ▶ Enforcing a minimum size leads to a power law tail
- ▶ With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- ▶ Take home message: Be careful out there...

Lognormals

Empirical Confusability

Random Multiplicative Growth Model

Random Growth with Variable Lifespan

References

Quick summary of these exciting developments

- ▶ Lognormals and power laws can be awfully similar
- ▶ Random Multiplicative Growth leads to lognormal distributions
- ▶ Enforcing a minimum size leads to a power law tail
- ▶ With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- ▶ Take home message: Be careful out there...

Lognormals


Empirical Confusability


Random Multiplicative Growth Model


Random Growth with Variable Lifespan

References

-  R. Axtell.
Zipf distribution of U.S. firm sizes.
Science, 293(5536):1818–1820, 2001. [pdf](#) (田)
-  R. Gibrat.
Les inégalités économiques.
Librairie du Recueil Sirey, Paris, France, 1931.
-  I. Gradshteyn and I. Ryzhik.
Table of Integrals, Series, and Products.
Academic Press, San Diego, fifth edition, 1994.
-  B. A. Huberman and L. A. Adamic.
Evolutionary dynamics of the World Wide Web.
Technical report, Xerox Palo Alto Research Center,
1999.

 B. A. Huberman and L. A. Adamic.
The nature of markets in the World Wide Web.
Quarterly Journal of Economic Commerce, 1:5–12,
2000.

 O. Malcai, O. Biham, and S. Solomon.
Power-law distributions and lévy-stable intermittent
fluctuations in stochastic systems of many
autocatalytic elements.
Phys. Rev. E, 60(2):1299–1303, Aug 1999. [pdf](#) (田)

 M. Mitzenmacher.
A brief history of generative models for power law and
lognormal distributions.
Internet Mathematics, 1:226–251, 2003. [pdf](#) (田)

Lognormals



Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

Lognormals

Empirical Confusability
Random Multiplicative
Growth Model
Random Growth with
Variable Lifespan

References

-  E. W. Montroll and M. W. Shlesinger.
On $1/f$ noise and other distributions with long tails.
Proc. Natl. Acad. Sci., 79:3380–3383, 1982.
-  E. W. Montroll and M. W. Shlesinger.
Maximum entropy formalism, fractals, scaling
phenomena, and $1/f$ noise: a tale of tails.
J. Stat. Phys., 32:209–230, 1983.