Lognormals and friends

Principles of Complex Systems
Course CSYS/MATH 300, Fall, 2009

Prof. Peter Dodds

Dept. of Mathematics & Statistics
Center for Complex Systems :: Vermont Advanced Computing Center
University of Vermont

Outline

Alternative distributions

There are other heavy-tailed distributions:

1. Lognormal
2. Stretched exponential (Weibull)
3. ... (Gamma)

lognormals

The lognormal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

- $\ln x$ is distributed according to a normal distribution with mean μ and variance σ.
- Appears in economics and biology where growth increments are distributed normally.

lognormals

Standard form reveals the mean μ and variance σ^2 of the underlying normal distribution:

$$P(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

For lognormals:

$$\mu_{\text{lognormal}} = e^{\mu+\frac{1}{2}\sigma^2}, \qquad \text{median}_{\text{lognormal}} = e^{\mu},$$

$$\sigma_{\text{lognormal}} = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}, \qquad \text{mode}_{\text{lognormal}} = e^{\mu-\sigma^2}.$$

All moments of lognormals are finite.

Derivation from a normal distribution

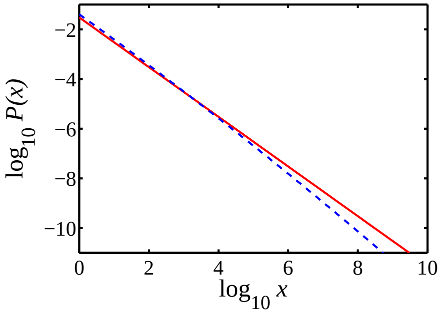
Take Y as distributed normally:

- $$P(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{d}y \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

Set $Y = \ln X$:

- Transform according to $P(x)\mathrm{d}x = P(y)\mathrm{d}y$:
- $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1/x \Rightarrow \mathrm{d}y = \mathrm{d}x/x$$
- $$\Rightarrow P(x)\mathrm{d}x = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x$$

Confusion between lognormals and pure power laws



Near agreement over four orders of magnitude!

- For lognormal (blue), $\mu = 0$ and $\sigma = 10$.
- For power law (red), $\gamma = 1$ and $c = 0.03$.

Confusion

What's happening:

- $$\ln P(x) = \ln\left\{\frac{1}{x\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\right\}$$
- $$= -\ln x - \ln\sqrt{2\pi} - \frac{(\ln x-\mu)^2}{2\sigma^2}$$
- $$= -\frac{1}{2\sigma^2}(\ln x)^2 + \left(\frac{\mu}{\sigma^2}-1\right)\ln x - \ln\sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}.$$
- $\Rightarrow$ If $\sigma^2 \gg 1$ and μ,
- $$\boxed{\ln P(x) \sim -\ln x + \text{const.}}$$

Confusion

- Expect -1 scaling to hold until $(\ln x)^2$ term becomes significant compared to $(\ln x)$.
- This happens when (roughly)
- $$-\frac{1}{2\sigma^2}(\ln x)^2 \simeq 0.05 \left(\frac{\mu}{\sigma^2} - 1\right) \ln x$$
- $$\Rightarrow \log_{10} x \lesssim 0.05 \times 2(\sigma^2 - \mu) \log_{10} e$$
- $$\simeq 0.05(\sigma^2 - \mu)$$
- $\Rightarrow$ If you find a -1 exponent,
  you may have a lognormal distribution...

Generating lognormals:

Random multiplicative growth:

- $$x_{n+1} = r x_n$$

  where $r > 0$ is a random growth variable
- (Shrinkage is allowed)
- In log space, growth is by addition:

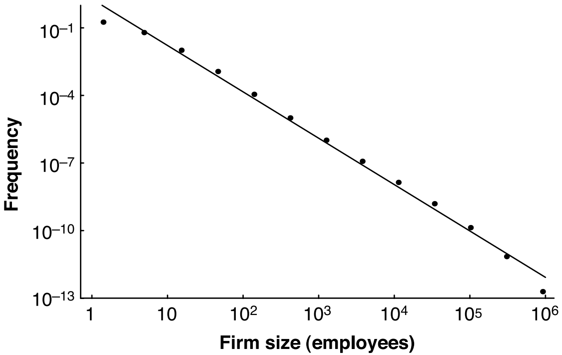
$$\ln x_{n+1} = \ln r + \ln x_n$$

- $\Rightarrow \ln x_n$ is normally distributed
- $\Rightarrow x_n$ is lognormally distributed

Lognormals or power laws?

- Gibrat [2] (1931) uses this argument to explain lognormal distribution of firm sizes
- Robert Axtell (2001) shows a power law fits the data very well [1]

$\gamma \simeq 2$



-

An explanation

- Axtel (mis)cites Malcai et al.'s (1999) argument [6] for why power laws appear with exponent $\gamma \simeq 1$
- The set up: N entities with size $x_i(t)$
- Generally:

  $$x_i(t+1) = r x_i(t)$$

  where r is drawn from some happy distribution
- Same as for lognormal but one extra piece:
- Each x_i cannot drop too low with respect to the other sizes:

  $$x_i(t+1) = \max(r x_i(t), c \langle x_i \rangle)$$

An explanation

Some math later...

Insert question 1, assignment 5 (⊞)

- $$\text{Find} \quad P(x) \sim x^{-\gamma}$$
- $$\text{where} \quad N = \frac{(\gamma-2)}{(\gamma-1)} \left[\frac{(c/N)^{\gamma-1} - 1}{(c/N)^{\gamma-1} - (c/N)} \right]$$
- $$\text{Now, if } c/N \ll 1, \quad N = \frac{(\gamma-2)}{(\gamma-1)} \left[\frac{-1}{-(c/N)} \right]$$
- $$\text{Which gives} \quad \gamma \sim 1 + \frac{1}{1-c}$$
- Groovy... c small $\Rightarrow \gamma \simeq 2$

The second tweak

Ages of firms/people/... may not be the same

- Allow the number of updates for each size x_i to vary
- Example: $P(t)\mathrm{d}t = ae^{-at}\mathrm{d}t$ where t = age.
- Back to no bottom limit: each x_i follows a lognormal
- Sizes are distributed as [7]

$$P(x) = \int_{t=0}^{\infty} ae^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2t}\right) \mathrm{d}t$$

(Assume for this example that $\sigma \sim t$ and $\mu = \ln m$)

- Now averaging different lognormal distributions.

Averaging lognormals

- $$P(x) = \int_{t=0}^{\infty} a e^{-at} \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\ln x/m)^2}{2t}\right) \mathrm{d}t$$
- Insert question 2, assignment 5 (⊞)
- Some enjoyable suffering leads to:

$$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$$

The second tweak

- $$P(x) \propto x^{-1} e^{-\sqrt{2\lambda(\ln x/m)^2}}$$
- Depends on sign of $\ln x/m$, i.e., whether $x/m > 1$ or $x/m < 1$.
- $$P(x) \propto \begin{cases} x^{-1+\sqrt{2\lambda}} & \text{if } x/m < 1 \\ x^{-1-\sqrt{2\lambda}} & \text{if } x/m > 1 \end{cases}$$
- 'Break' in scaling (not uncommon)
- Double-Pareto distribution (⊞)
- First noticed by Montroll and Shlesinger [8, 9]
- Later: Huberman and Adamic [4, 5]: Number of pages per website

Quick summary of these exciting developments

- Lognormals and power laws can be awfully similar
- Random Multiplicative Growth leads to lognormal distributions
- Enforcing a minimum size leads to a power law tail
- With no minimum size but a distribution of lifetimes, the double Pareto distribution appears
- Take home message: Be careful out there...

References I

R. Axtell.
Zipf distribution of U.S. firm sizes.
Science, 293(5536):1818–1820, 2001. pdf (⊞)

R. Gibrat.
Les inégalités économiques.
Librairie du Recueil Sirey, Paris, France, 1931.

I. Gradshteyn and I. Ryzhik.
Table of Integrals, Series, and Products.
Academic Press, San Diego, fifth edition, 1994.

B. A. Huberman and L. A. Adamic.
Evolutionary dynamics of the World Wide Web.
Technical report, Xerox Palo Alto Research Center, 1999.

References II

B. A. Huberman and L. A. Adamic.
The nature of markets in the World Wide Web.
Quarterly Journal of Economic Commerce, 1:5–12, 2000.

O. Malcai, O. Biham, and S. Solomon.
Power-law distributions and lévy-stable intermittent fluctuations in stochastic systems of many autocatalytic elements.
Phys. Rev. E, 60(2):1299–1303, Aug 1999. pdf (⊞)

M. Mitzenmacher.
A brief history of generative models for power law and lognormal distributions.
Internet Mathematics, 1:226–251, 2003. pdf (⊞)

References III

E. W. Montroll and M. W. Shlesinger.
On $1/f$ noise aned other distributions with long tails.
Proc. Natl. Acad. Sci., 79:3380–3383, 1982.

E. W. Montroll and M. W. Shlesinger.
Maximum entropy formalism, fractals, scaling phenomena, and $1/f$ noise: a tale of tails.
J. Stat. Phys., 32:209–230, 1983.