CSYS/MATH 300: Principles of Complex Systems—Assignment 4
University of Vermont, Fall 2009

Dispersed: Monday, November 16, 2009.
Due: By start of lecture, 10:00 am, Tuesday, November 24, 2009.
Office hours: 11:00 am to 2 pm, Wednesday, Farrell Hall.
Course website: `http://www.uvm.edu/~pdodds/teaching/courses/2009-08UVM-300/`

All questions are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated (fellow students, software programs, adversaries, etc.).

1. *Zipfarama via Optimization:*

   Complete the Mandelbrotian derivation of Zipf's law by minimizing the function

   $$\Psi(p_1, p_2, \ldots, p_n) = F(p_1, p_2, \ldots, p_n) + \lambda G(p_1, p_2, \ldots, p_n)$$

   where the 'cost over information' function is

   $$F(p_1, p_2, \ldots, p_n) = \frac{C}{H} = \frac{\sum_{i=1}^{n} p_i \ln(i+a)}{-g \sum_{i=1}^{n} p_i \ln p_i}$$

   and the constraint function is

   $$G(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} p_i - 1 = 0$$

   to find

   $$p_j = (j+a)^{-\alpha}$$

   where $\alpha = H/gC$.

   Note: We have now allowed the cost factor to be $(j+a)$ rather than $(j+1)$. Exciting!

   Hint: when finding λ, find an expression connecting λ, g, C, and H. Extra hint: one way may be substitute the form you find for $\ln p_i$ into H's definition (but do not replace p_i).

2. (a) For $n \to \infty$, use some computation tool (e.g., Matlab) to determine that $\alpha \simeq 1.73$ for $a = 1$. (Recall: we expect $\alpha < 1$.)

   (b) For finite n, find an approximate estimate of a in terms of n that yields $\alpha = 1$. (Hint: use an integral approximation for the relevant sum.)

   (c) What happens to a as $n \to \infty$?