CSYS/MATH 300: Principles of Complex Systems—Assignment 1
University of Vermont, Fall 2009

Dispersed: Friday, October 16, 2009.
Due: By start of lecture, 10:00 am, Thursday, October 29, 2009.
Sections covered: .

Some useful reminders:
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All questions are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated (fellow students, software programs, adversaries, etc.).

1. (3 pts for a-d, 3 pts for e-g)

   Consider a random variable X with a probability distribution given by

   $$P(x) = cx^{-\gamma}$$

   where c is a normalization constant, and $0 < a \le x \le b$. (a and b are the lower and upper cutoffs respectively.) Assume that $\gamma > 1$.

   (a) Determine c.

   (b) Compute the nth moment of X.

   (c) In the limit $b \to \infty$, how does the nth moment behave as a function of γ?

   (d) For finite cutoffs a and b (still with $a \ll b$), which cutoff dominates the expressions for the moments as a function of γ? (Note: both may be involved to some degree.)

   (e) Noting what constraints, if any, we must place on γ for the mean to be finite in the case $b \to \infty$, now find σ, the standard deviation of X.

   How does σ behave as a function of γ?

   (f) Compute the mean absolute displacement (MAD).

   The mean absolute displacement is given by $\langle |X - \langle X \rangle| \rangle$ where $\langle \cdot \rangle$ represents expected value.

   How does MAD behave as a function of γ? How does this compare with the variance?

(g) Why did we assume $\gamma > 1$?

2. Consider a modified version of the Barabàsi-Albert (BA) model [1] where two possible mechanisms are now in play. As in the original model, start with m_0 nodes at time $t = 0$. Let's make these initial guys connected such that each has degree 1. The two mechanisms are:

   M1: With probability p, a new node of degree 1 is added to the network. At time $t+1$, a node connects to an existing node j with probability

   $$P(\text{connect to node } j) = \frac{k_j}{\sum_{i=1}^{N(t)} k_i} \tag{1}$$

   where k_j is the degree of node j and $N(t)$ is the number of nodes in the system at time t.

   M2: With probability $q = 1 - p$, a randomly chosen node adds a new edge, connecting to node j with the same preferential attachment probability as above.

   Note that in the limit $q = 0$, we retrieve the original BA model (with the difference that we are adding one link at a time rather than m here).

   In the long time limit $t \to \infty$, what is the expected form of the degree distribution P_k?

3. Now take the Barabàsi-Albert model with an attachment kernel $A_k = k^{1/2}$. Take newly arriving nodes as adding m links ($m = 1$ for the preceding question).

   Use the same approach as in class (which is a modified version of the original derivation in [1]), to determine the long-time limiting form of the degree distribution P_k.

   A catch and a hint: to normalize the attachment kernel at each point in time t, we have to divide by the sum of all degrees in the network (as per Eq. 1 above). Recall that for the original model, the sum of all degrees nicely simplified to $2mt + m_0$ (check over this). But now we have the sum of $k_i^{1/2}$, and its form is not obvious. Here's the help: assume that

   $$\sum_{i=1}^{N(t)} k_i^{1/2} = \lambda t$$

   where λ is to be determined later. In other words, assume that the normalization factor grows linearly with t, as it did for the original model. If this is indeed true, then you will be able to justify it once you have found P_k.

References

[1] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *Science*, 286:509–511, 1999.