Scale-Free Networks Complex Networks, Course 303A, Spring, 2009

Prof. Peter Dodds

Department of Mathematics & Statistics University of Vermont



Licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 License.

Original model

Model deta

Analysis

nechanism

Robustness

Redner & Krapivisky's mode

Generalized model

Analysis

Universality?

Sublinear attachme

Superlinear attac

References

References



Outline

Original model

Introduction

Model details

Analysis

A more plausible mechanism

Robustness

Redner & Krapivisky's model

Generalized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachment kernels

References

Original model

Model det

A more plausit nechanism

Robustness

Redner & Krapivisky's mode

Analysis
Universality?

Universality? Sublinear attachr

Superlinear attach

References



Outline

Original model

Introduction

Model details

Analysis

A more plausible mechanism

Generalized model

Analysis

Superlinear attachment kernels

Original model Introduction

Redner &

References

Frame 3/57





Scale-free networks

- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree

- One of the seminal works in complex networks:
- Somewhat misleading nomenclature...

Original model Introduction

Redner &

References



- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma}$$
 for 'large' k

- One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999: "Emergence of scaling in random networks" [2]
- Somewhat misleading nomenclature...

Original model
Introduction
Model details
Analysis

Redner & Krapivisky's model

eneralized model lalysis niversality?

kernels Superlinear attach

References



- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma}$$
 for 'large' k

- One of the seminal works in complex networks:
 Laszlo Barabási and Reka Albert, Science, 1999:
 "Emergence of scaling in random networks" [2]
- Somewhat misleading nomenclature...

Original model
Introduction
Model details
Analysis

Redner & Krapivisky's model

Analysis

Universality?

kernels Superlinear attac

Superlinear attach kernels

References



- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma}$$
 for 'large' k

- One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999: "Emergence of scaling in random networks" [2]
- Somewhat misleading nomenclature...

Original model Introduction

Redner &

References



- Networks with power-law degree distributions have become known as scale-free networks.
- Scale-free refers specifically to the degree distribution having a power-law decay in its tail:

$$P_k \sim k^{-\gamma}$$
 for 'large' k

- One of the seminal works in complex networks: Laszlo Barabási and Reka Albert, Science, 1999: "Emergence of scaling in random networks" [2]
- Somewhat misleading nomenclature...

Original model Introduction

Redner &

References



Scale-free networks

- Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are

Original model Introduction

Redner &

References



Scale-free networks

- Scale-free networks are not fractal in any sense.
- Usually talking about networks whose links are abstract, relational, informational, ... (non-physical)
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not...

Original model

Introduction
Model details
Analysis
A more plausible
mechanism

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

Superlinear attachmikernels

References



Scale-free networks are not fractal in any sense.

abstract, relational, informational, ... (non-physical)

Usually talking about networks whose links are

Primary example: hyperlink network of the Web Much arguing about whether or networks are

References



References

Scale-free networks are not fractal in any sense.

- Usually talking about networks whose links are abstract, relational, informational, ... (non-physical)
- Primary example: hyperlink network of the Web
- Much arguing about whether or networks are 'scale-free' or not



Original model

References

Introduction

Redner &

Frame 6/57





 $\gamma = 2.5$

 $\langle k \rangle = 1.8$



 $\gamma = 2.5$

 $\langle k \rangle = 2.05333$











 $\gamma = 2.5$

 $\langle k \rangle = 1.8$

 $\gamma = 2.5$

 $\langle k \rangle = 1.92$





$$\gamma = 2.5$$
 $\gamma = 2.5$ $\langle k \rangle = 1.6$ $\langle k \rangle = 1.50$



The big deal:

We move beyond describing of networks to finding mechanisms for why certain networks are the way they are.

A big deal for scale-free networks:

- How does the exponent γ depend on the mechanism?
- Do the mechanism details matter?

Original model

Introduction Model details Analysis A more plausible

Robustness
Redner &

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

Superlinear attachme

References

Frame 7/57



Scale-free networks

The big deal:

We move beyond describing of networks to finding mechanisms for why certain networks are the way they are.

A big deal for scale-free networks:

- \blacktriangleright How does the exponent γ depend on the mechanism?
- Do the mechanism details matter?

Original model Introduction

Redner &

References

Frame 7/57



Scale-free networks

The big deal:

We move beyond describing of networks to finding mechanisms for why certain networks are the way they are.

A big deal for scale-free networks:

- ▶ How does the exponent γ depend on the mechanism?
- Do the mechanism details matter?

Original model

Introduction

Model details

Analysis

A more plausible mechanism

Redner & Kranivisky's model

Analysis
Universality?
Sublinear attachment
kernels
Superlinear attachment

References

Frame 7/57



Work that presaged scale-free networks

- ▶ 1924: G. Udny Yule [9]: # Species per Genus
- ▶ 1926: Lotka ^[4]: # Scientific papers per author
- ▶ 1953: Mandelbrot ^[5]): Zipf's law for word frequency through optimization
- 1955: Herbert Simon [8, 10]: Zipf's law, city size, income, publications, and species per genus
- ▶ 1965/1976: Derek de Solla Price [6, 7]: Network of Scientific Citations

Original model

Model details Analysis A more plausible mechanism

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment
kernels

References



Work that presaged scale-free networks

- ▶ 1924: G. Udny Yule [9]: # Species per Genus
- ▶ 1926: Lotka [4]: # Scientific papers per author
- ▶ 1953: Mandelbrot ^[5]): Zipf's law for word frequency through optimization
- 1955: Herbert Simon [8, 10]: Zipf's law, city size, income, publications, and species per genus
- ▶ 1965/1976: Derek de Solla Price [6, 7]: Network of Scientific Citations

Original model
Introduction

Model details Analysis A more plausible

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

References



Work that presaged scale-free networks

- 1924: G. Udny Yule [9]: # Species per Genus
- ▶ 1926: Lotka [4]: # Scientific papers per author
- ▶ 1953: Mandelbrot [5]): Zipf's law for word frequency through optimization
- ▶ 1955: Herbert Simon [8, 10]:
- ▶ 1965/1976: Derek de Solla Price [6, 7]:

Original model Introduction

Redner &

References



Work that presaged scale-free networks

- ▶ 1924: G. Udny Yule [9]: # Species per Genus
- ▶ 1926: Lotka ^[4]: # Scientific papers per author
- ▶ 1953: Mandelbrot ^[5]): Zipf's law for word frequency through optimization
- ▶ 1955: Herbert Simon [8, 10]: Zipf's law, city size, income, publications, and species per genus
- ▶ 1965/1976: Derek de Solla Price [6, 7]: Network of Scientific Citations

Original model

Model detail: Analysis

> A more plausit nechanism Robustness

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

References



Work that presaged scale-free networks

- 1924: G. Udny Yule [9]: # Species per Genus
- 1926: Lotka [4]: # Scientific papers per author
- ▶ 1953: Mandelbrot [5]): Zipf's law for word frequency through optimization
- ▶ 1955: Herbert Simon [8, 10]: Zipf's law, city size, income, publications, and species per genus
- ▶ 1965/1976: Derek de Solla Price [6, 7]: Network of Scientific Citations

Original model Introduction

Redner &

References



Outline

Original model

Introduction

Model details

Analysis

A more plausible mechanism

Redner & Krapivisky's model

Generalized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachment kernels

References

Original model

Model details

Analysis A more plaus

Robustness

Redner & Krapivisky's mode

Analysis

Universality:

kernels

Superlinear attach

References

Frame 9/57



- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA)
- ▶ Step 1: start with m_0 disconnected nodes.
- ► Step 2:
 - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
 - Each new node makes m links to nodes already present.
 - Preferential attachment—Probability of connecting to ith node is ∝ k_i.
- ▶ In essence, we have a rich-gets-richer scheme.

Original model

Model details

Analysis
A more plausil

Redner & Kranivisky's mode

Generalized model Analysis Universality?

Sublinear attachmen ernels

ernels

References





- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- Step 1: start with m₀ disconnected nodes
- ► Step 2:
 - 1. Growth—a new node appears at each time step t = 0.1.2
 - Each new node makes m links to nodes already present.
 - 3. Preferential attachment—Probability of connecting to *i*th node is $\propto k_0$.
- ▶ In essence, we have a rich-gets-richer scheme.

Original model

Model details

Analysis
A more plausi

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment

uperlinear attachment ernels

References



- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- ▶ Step 1: start with *m*₀ disconnected nodes.
- ► Step 2:
 - 1. Growth—a new node appears at each time step $t = 0.1, 2, \dots$
 - Each new node makes m links to nodes already present.
 - 3. Preferential attachment—Probability of connecting to *i*th node is $\propto k_0$.
- ▶ In essence, we have a rich-gets-richer scheme.

Original model

Model details

Analysis A more plaus mechanism

Redner & Krapivisky's mode

Analysis
Universality?

ernels Superlinear attachment

.....

References





- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- ▶ Step 1: start with *m*₀ disconnected nodes.
- ▶ Step 2:
 - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
 - Each new node makes m links to nodes already present.
 - 3. Preferential attachment—Probability of connecting to *i*th node is $\propto k_i$.
- ▶ In essence, we have a rich-gets-richer scheme.

Original model

Model details

Analysis A more plausil mechanism

Redner & Krapivisky's mode

Generalized model
Analysis
Universality?
Sublinear attachment

ernels uperlinear attachment

References



- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- ▶ Step 1: start with *m*₀ disconnected nodes.
- ▶ Step 2:
 - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
 - Each new node makes m links to nodes already present.
 - 3. Preferential attachment—Probability of connecting to ith node is $\propto k_i$.
- ▶ In essence, we have a rich-gets-richer scheme.

Original model

Model details

Anarysis A more plausib mechanism Robustnoss

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachmen

uperlinear attachment

References



- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- ▶ Step 1: start with m_0 disconnected nodes.
- ▶ Step 2:
 - Growth—a new node appears at each time step t = 0, 1, 2,
 - Each new node makes m links to nodes already present.
 - 3. Preferential attachment—Probability of connecting to *i*th node is $\propto k_i$.
- ▶ In essence, we have a rich-gets-richer scheme.

Original model

Model details

Analysis
A more plausil

Redner & Krapivisky's mode

Generalized model
Analysis
Universality?
Sublinear attachment

References



- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- ▶ Step 1: start with *m*₀ disconnected nodes.
- ▶ Step 2:
 - 1. Growth—a new node appears at each time step t = 0, 1, 2, ...
 - Each new node makes m links to nodes already present.
 - 3. Preferential attachment—Probability of connecting to ith node is $\propto k_i$.
- ▶ In essence, we have a rich-gets-richer scheme.

Original model

Model details

Analysis A more plausib mechanism

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachmer

erneis uperlinear attachme ernels

References



- Barabási-Albert model = BA model.
- Key ingredients: Growth and Preferential Attachment (PA).
- ▶ Step 1: start with *m*₀ disconnected nodes.
- ▶ Step 2:
 - Growth—a new node appears at each time step t = 0, 1, 2,
 - Each new node makes m links to nodes already present.
 - 3. Preferential attachment—Probability of connecting to ith node is $\propto k_i$.
- In essence, we have a rich-gets-richer scheme.

Original model

Model details Analysis

> A more plausit mechanism Robustness

Redner & Krapivisky's mode

Generalized model Analysis Universality? Sublinear attachment

rnels

References





Outline

Original model

Model details

Analysis

A more plausible mechanism

Generalized model

Analysis

Superlinear attachment kernels

Original model

Analysis

Redner &

References

Frame 11/57





- ▶ Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- ▶ Definition: $P_{\text{attach}}(k, t)$ is the attachment probability.
- ► For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=m}^{k_{\text{max}}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

Original model

Model details Analysis

. more plausibl nechanism lobustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

ernels Superlinear attachr

.....

References



- ▶ Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- ▶ Definition: $P_{\text{attach}}(k, t)$ is the attachment probability.
- ► For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=m}^{k_{\text{max}}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

Original model

Model details

Analysis

A more plausit nechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

ernels uperlinear attach

References



- ▶ Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- **Definition:** $P_{\text{attach}}(k, t)$ is the attachment probability.
- ► For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=m}^{k_{\text{max}}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

Original model

Model details

Analysis

A more plausi

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment

uperlinear attachment ernels

References



- ▶ Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- **Definition:** $P_{\text{attach}}(k, t)$ is the attachment probability.
- ► For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=m}^{k_{\text{max}}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

Original model

Model details Analysis

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachmen

perlinear attachme

References



- ▶ Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- **Definition:** $P_{\text{attach}}(k, t)$ is the attachment probability.
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=m}^{k_{\text{max}}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

Original model

Model details Analysis

A more plausit nechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

uperlinear attachme ernels

References



BA model

- ▶ Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- ▶ Definition: $P_{\text{attach}}(k, t)$ is the attachment probability.
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=m}^{k_{\text{max}}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

Original model

Model details Analysis

A more plausit nechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

uperlinear attach ernels

References

Frame 12/57



BA model

- ▶ Definition: A_k is the attachment kernel for a node with degree k.
- For the original model:

$$A_k = k$$

- ▶ Definition: $P_{\text{attach}}(k, t)$ is the attachment probability.
- For the original model:

$$P_{\text{attach}}(\text{node } i, t) = \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = \frac{k_i(t)}{\sum_{k=m}^{k_{\text{max}}(t)} k N_k(t)}$$

where $N(t) = m_0 + t$ is # nodes at time t and $N_k(t)$ is # degree k nodes at time t.

Original model

Model details Analysis

A more plausit nechanism

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment
kernels

References

Frame 12/57



Approximate analysis

▶ When (N + 1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1}-k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- ▶ Approximate $k_{i,N+1} k_{i,N}$ with $\frac{d}{dt}k_{i,t}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where $t = N(t) - m_0$.

Original model

Model details

Analysis

A more plaus

nechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment
kernels

References



Approximate analysis

▶ When (N + 1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1}-k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- ▶ Approximate $k_{i,N+1} k_{i,N}$ with $\frac{d}{dt}k_{i,t}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m \frac{\kappa_i(t)}{\sum_{j=1}^{N(t)} \kappa_j(t)}$$

where $t = N(t) - m_0$.

Original model

Introduction Model details Analysis

Analysis A more plaus

Robustness

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment
kernels

Superlinear attachmer ernels

References



 \blacktriangleright When (N+1)th node is added, the expected increase in the degree of node *i* is

$$E(k_{i,N+1}-k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- Approximate $k_{i,N+1} k_{i,N}$ with $\frac{d}{dt}k_{i,t}$:

Original model

Analysis

Redner &

References



When (N + 1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1}-k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- ▶ Approximate $k_{i,N+1} k_{i,N}$ with $\frac{d}{dt}k_{i,t}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m \frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)}$$

where
$$t = N(t) - m_0$$
.

Original model

Model details Analysis

Analysis
A more plausil

Redner &

Generalized model
Analysis
Universality?
Sublinear attachment

.....

References



▶ When (N + 1)th node is added, the expected increase in the degree of node i is

$$E(k_{i,N+1}-k_{i,N}) \simeq m \frac{k_{i,N}}{\sum_{j=1}^{N(t)} k_j(t)}.$$

- Assumes probability of being connected to is small.
- Dispense with Expectation by assuming (hoping) that over longer time frames, degree growth will be smooth and stable.
- ▶ Approximate $k_{i,N+1} k_{i,N}$ with $\frac{d}{dt}k_{i,t}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t}=m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)}$$

where $t = N(t) - m_0$.

Original model

Model detail

A more plausit nechanism

Redner & Kranivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

References



$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

► The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow k_i(t) = c_i t^{1/2}.$$

ightharpoonup Next find $c_i \dots$

Original model

Introduction Model details

Analysis

mechanism Robustness

Redner & Krapivisky's model

Analysis

Jniversality?

ernels

Superlinear attach kernels

References



Approximate analysis

▶ Deal with denominator: each added node brings m new edges.

$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

► The node degree equation now simplifies:

$$\frac{d}{dt}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)} k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \left[k_i(t) = c_i t^{1/2}\right].$$

ightharpoonup Next find $c_i \dots$

Original model

Model details

Analysis

mechanism Rebustness

Redner & Krapivisky's model

Generalized mo Analysis

Universality?

ernels

Superlinear attach kernels

References



$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

► The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \left[k_i(t) = c_i t^{1/2}\right]$$

ightharpoonup Next find $c_i \dots$

Original model

Model details

Analysis

mechanism Robustness

Redner & Krapivisky's mode

aeneralized mo unalysis

niversality?

Sublinear attachment ernels

Superlinear attachmer

References



$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow k_i(t) = c_i t^{1/2}.$$

 \triangleright Next find $c_i \dots$

Original model

Model detail

Analysis

mechanism Robustness

Redner & Krapivisky's mode

Generalized mi Analysis

Iniversality?

Sublinear attachment sernels

Superlinear attach kernels

References



$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow k_i(t) = c_i t^{1/2}.$$

ightharpoonup Next find $c_i \dots$

Original model

Model detail

Analysis

mechanism Robustness

Redner & Krapivisky's model

nalysis

niversality?

Sublinear attachment ternels

uperlinear attachm

References



$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

► The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i t^{1/2}}.$$

ightharpoonup Next find $c_i \dots$

Original model

Model detail

Analysis

mechanism Robustness

Redner & Krapivisky's mode

Analysis Jniversality?

ernels uperlinear attachment

References



$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

► The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i t^{1/2}}.$$

ightharpoonup Next find $c_i \dots$

Original model

Model detail

Analysis

mechanism Robustness

Redner & Krapivisky's mode

> eneralized mode nalysis

niversality?

Superlinear attachmer ernels

References



$$\therefore \sum_{j=1}^{N(t)} k_j(t) = 2tm$$

► The node degree equation now simplifies:

$$\frac{\mathrm{d}}{\mathrm{d}t}k_{i,t} = m\frac{k_i(t)}{\sum_{j=1}^{N(t)}k_j(t)} = m\frac{k_i(t)}{2mt} = \frac{1}{2t}k_i(t)$$

Rearrange and solve:

$$\frac{\mathrm{d}k_i(t)}{k_i(t)} = \frac{\mathrm{d}t}{2t} \Rightarrow \boxed{k_i(t) = c_i t^{1/2}}.$$

ightharpoonup Next find $c_i \dots$

Original model

Model detail

Analysis A more plaus

mechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

ernels iuperlinear attachment ernels

References



$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

▶ So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \text{ for } t \geq t_{i,\text{start}}.$$

- ► All node degrees grow as t^{1/2} but later nodes have larger t_{i,start} which flattens out growth curve.
- ► Early nodes do best (First-mover advantage).

Original model

Introduction Model details

Analysis
A more plausil

mechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

ernels uperlinear attachment

References

Frame 15/57



$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

▶ So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \text{ for } t \geq t_{i,\text{start}}.$$

- ► All node degrees grow as t^{1/2} but later nodes have larger t_{i,start} which flattens out growth curve.
- ► Early nodes do best (First-mover advantage).

Original model

Model detail

Analysis A more plausi

Redner &

Analysis
Universality?
Sublinear attachment

ernels uperlinear attachment

References

Frame 15/57



$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

▶ So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \text{ for } t \geq t_{i,\text{start}}.$$

- ➤ All node degrees grow as t^{1/2} but later nodes have larger t_{i,start} which flattens out growth curve.
- Early nodes do best (First-mover advantage).

Original model

Model detail

Analysis A more plaus

Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachmen

rnels uperlinear attachment

References

Frame 15/57



$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

▶ So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \text{ for } t \geq t_{i,\text{start}}.$$

- All node degrees grow as $t^{1/2}$ but later nodes have larger $t_{i,\text{start}}$ which flattens out growth curve.
- Early nodes do best (First-mover advantage).

Original model

Introduction Model details

Analysis A more plaus

mechanism Robustness

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment

ernels Superlinear attachment

References

Frame 15/<u>57</u>



$$t_{i,\text{start}} = \begin{cases} i - m_0 & \text{for } i > m_0 \\ 0 & \text{for } i \le m_0 \end{cases}$$

▶ So for $i > m_0$ (exclude initial nodes), we must have

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \text{ for } t \geq t_{i,\text{start}}.$$

- All node degrees grow as t^{1/2} but later nodes have larger $t_{i,start}$ which flattens out growth curve.
- Early nodes do best (First-mover advantage).

Original model

Analysis

Redner &

References





Analysis

Redner &

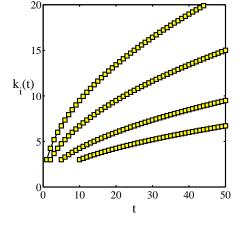
m = 3 $ightharpoonup t_{i,start} =$

1, 2, 5, and 10.

References

Frame 16/57





- ▶ So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly:

$$P(t_{i,\text{start}}) dt_{i,\text{start}} \simeq \frac{dt_{i,\text{start}}}{t + m_0}$$

Using

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

and by understanding that later arriving nodes have lower degrees, we can say this:

$$\mathbf{Pr}(k_i < k) = \mathbf{Pr}(t_{i,\text{start}} > \frac{m^2 t}{k^2})$$

Original model

Model details

Analysis

A more plausible

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment kernels

References



- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly:

$$P(t_{i,\text{start}}) dt_{i,\text{start}} \simeq \frac{dt_{i,\text{start}}}{t + m_0}$$

Using

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

and by understanding that later arriving nodes have lower degrees, we can say this:

$$\mathbf{Pr}(k_i < k) = \mathbf{Pr}(t_{i,\text{start}} > \frac{m^2 t}{k^2})$$

Original model

Model details Analysis

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment
kernels

References



- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly:

$$P(t_{i,\text{start}}) dt_{i,\text{start}} \simeq \frac{dt_{i,\text{start}}}{t + m_0}$$

Using

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

and by understanding that later arriving nodes have lower degrees, we can say this:

$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}).$$

Original model

Introduction Model detai

Analysis
A more plausit

Redner & Kranivisky's model

Analysis
Universality?
Sublinear attachment

uperlinear attachmen ernels

References



- So what's the degree distribution at time t?
- Use fact that birth time for added nodes is distributed uniformly:

$$P(t_{i,\text{start}}) dt_{i,\text{start}} \simeq \frac{dt_{i,\text{start}}}{t + m_0}$$

Using

$$k_i(t) = m \left(\frac{t}{t_{i,\text{start}}}\right)^{1/2} \Rightarrow t_{i,\text{start}} = \frac{m^2 t}{k_i(t)^2}.$$

and by understanding that later arriving nodes have lower degrees, we can say this:

$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}).$$

Original model

Introduction Model detail Analysis

> A more plausit nechanism Robustness

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment kernels

References



$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

▶ Differentiate to find Pr(k):

$$Pr(k) = \frac{d}{dk}Pr(k_i < k) = \frac{2m^2t}{(t+m_0)k^3}$$

 $\sim 2m^2k^{-3}$ as $m\to\infty$.

Original model

Introduction Model detail

Analysis

mechanism Robustness

Redner & Krapivisky's mode

> nalysis niversality?

Sublinear attachment sernels

Superlinear attachm ernels

References



$$\mathbf{Pr}(k_i < k) = \mathbf{Pr}(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

Differentiate to find Pr(k):

$$\mathbf{Pr}(k) = \frac{\mathrm{d}}{\mathrm{d}k} \mathbf{Pr}(k_i < k) = \frac{2m^2t}{(t + m_0)k^3}$$

$$\sim 2m^2k^{-3}$$
 as $m\to\infty$

Original model

Model detail

Analysis A more plaus

mechanism Robustness

Redner & Krapivisky's model

Analysis

Jniversality? Sublinear attachn

Sublinear attachment kernels Superlinear attachment

References



$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

Differentiate to find Pr(k):

$$\Pr(k) = \frac{d}{dk} \Pr(k_i < k) = \frac{2m^2t}{(t + m_0)k^3}$$

$$\sim 2m^2k^{-3}$$
 as $m\to\infty$

Original model

Introduction Model detail

Analysis

a more plausit nechanism Robustness

Redner & Krapivisky's model

Analysis Universality? Sublinear attachment

ernels Superlinear attachment

References

References



$$\Pr(k_i < k) = \Pr(t_{i,\text{start}} > \frac{m^2 t}{k^2}) \simeq \frac{t - \frac{m^2 t}{k^2}}{t + m_0}.$$

Differentiate to find Pr(k):

$$\mathbf{Pr}(k) = \frac{\mathrm{d}}{\mathrm{d}k} \mathbf{Pr}(k_i < k) = \frac{2m^2t}{(t+m_0)k^3}$$

$$\sim 2m^2k^{-3}$$
 as $m\to\infty$.

Original model

Introduction Model detail

Analysis A more plaus

nechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?

ernels uperlinear attachment

References



- ▶ We thus have a very specific prediction of $Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- ▶ 2 < γ < 3: finite mean and 'infinite' variance (wild)
- In practice, γ < 3 means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$: finite mean and variance (mild)

Model detail

Analysis

A more plausib nechanism

Redner &

Generalized model
Analysis
Universality?
Sublinear attachment

kerneis Superlinear attach

References

References





- We thus have a very specific prediction of $Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.
- ightharpoonup 2 < γ < 3: finite mean and 'infinite' variance (wild)
- In practice, $\gamma < 3$ means variance is governed by
- $ightharpoonup \gamma > 3$: finite mean and variance (mild)

Analysis

Redner &

References



- We thus have a very specific prediction of $Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- ightharpoonup 2 < γ < 3: finite mean and 'infinite' variance (wild)
- In practice, $\gamma < 3$ means variance is governed by
- $ightharpoonup \gamma > 3$: finite mean and variance (mild)

Analysis

Redner &

References



- ▶ We thus have a very specific prediction of $Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- **2** < γ < **3**: finite mean and 'infinite' variance (wild)
- In practice, γ < 3 means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$: finite mean and variance (mild)

Original model

Introduction Model detail Analysis

A more plaus

Robustness

Redner & Krapivisky's mode

Analysis
Universality?

kernels

Superlinear atta

References



- We thus have a very specific prediction of $Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- ightharpoonup 2 < γ < 3: finite mean and 'infinite' variance (wild)
- In practice, γ < 3 means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$: finite mean and variance (mild)

Analysis

Redner &

References



- ▶ We thus have a very specific prediction of $Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- **2** < γ < **3**: finite mean and 'infinite' variance (wild)
- In practice, γ < 3 means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$: finite mean and variance (mild)

Introduction Model details Analysis

mechanism Robustness

Redner & Krapivisky's mode

Generalized model Analysis Universality? Sublinear attachment kernels

ernels

References



- ▶ Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- $ightharpoonup 2 < \gamma < 3$: finite mean and 'infinite' variance (wild)
- In practice, γ < 3 means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$: finite mean and variance (mild)

Analysis

Redner &

References



- ▶ We thus have a very specific prediction of $Pr(k) \sim k^{-\gamma}$ with $\gamma = 3$.
- ▶ Typical for real networks: $2 < \gamma < 3$.
- Range true more generally for events with size distributions that have power-law tails.
- **2** < γ < **3**: finite mean and 'infinite' variance (wild)
- In practice, γ < 3 means variance is governed by upper cutoff.
- $ightharpoonup \gamma > 3$: finite mean and variance (mild)

Original model

Introduction Model details Analysis

A more plaus mechanism

Robustness
Redner &

Krapivisky's model
Generalized model
Analysis

kernels
Superlinear attach

. .

References

Frame 19/57



Analysis

Redner &

References

Frame 20/57





WWW $\gamma \simeq$ 2.1 for in-degree WWW $\gamma \simeq 2.45$ for out-degree Movie actors $\gamma \simeq 2.3$ Words (synonyms) $\gamma \simeq 2.8$

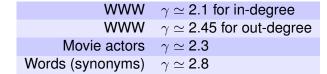
Redner &

References

Frame 20/57







The Internets is a different business...

Real data

From Barabási and Albert's original paper [2]:

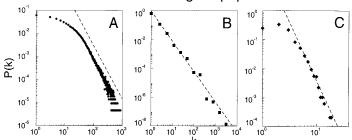


Fig. 1. The distribution function of connectivities for various large networks. (A) Actor collaboration graph with N=212,250 vertices and average connectivity $\langle k \rangle=28.78$. (B) WWW, N=325,729, $\langle k \rangle=5.46$ (6). (C) Power grid data, N=4941, $\langle k \rangle=2.67$. The dashed lines have slopes (A) $\gamma_{\rm actor}=2.3$, (B) $\gamma_{\rm www}=2.1$ and (C) $\gamma_{\rm power}=4$.

Original model

Introductio Model deta

> Analysis A more plausi

Redner &

Generalized model
Analysis
Universality?
Sublinear attachment

_ .

References

Frame 21/57



- Vary attachment kernel.
- Vary mechanisms:
 - Add edge deletion
 - Add node deletion
 - Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- ightharpoonup Q.: How does changing the model affect γ ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter?
- ► The answer is (surprisingly) yes.

Original model

Model details

A more plausil mechanism

Robustness
Redner &

Redner & Krapivisky's model

Analysis
Universality?

Sublinear attachme

Superlinear attach

References



- Vary attachment kernel.
- Vary mechanisms:
 - 1. Add edge deletion
 - 2. Add node deletion
 - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- ightharpoonup Q.: How does changing the model affect γ ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter?
- ► The answer is (surprisingly) yes.

Original model

Model details
Analysis

mechanism Robustness

Redner & Krapivisky's model

Analysis Universality? Sublinear attachmer

kernels Superlinear attachmen kernels

References



- Vary attachment kernel.
- Vary mechanisms:
 - Add edge deletion
 - Add node deletion
 - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- ightharpoonup Q.: How does changing the model affect γ ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter?
- ► The answer is (surprisingly) yes.

Original model

Model details

Analysis

A more plausil mechanism Robustness

Redner & Krapivisky's model

Analysis Universality? Sublinear attachmer

ærnels Superlinear attachmer ærnels

References

References



- Vary attachment kernel.
- Vary mechanisms:
 - Add edge deletion
 - Add node deletion
- Deal with directed versus undirected networks.
- \triangleright Q.: How does changing the model affect γ ?
- Q.: Do we need preferential attachment and growth?
- Q: Do model details matter?
- ► The answer is (surprisingly) yes.

Original model

Analysis

Redner &

References



- Vary attachment kernel.
- Vary mechanisms:
 - Add edge deletion
 - 2. Add node deletion
 - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- ightharpoonup Q.: How does changing the model affect γ ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter?
- ► The answer is (surprisingly) yes.

Original model

Model details

Analysis A more plausil mechanism

Redner &

Krapivisky's model
Generalized model
Apalusia

Universality? Sublinear attachmen kernels

kerneis Superlinear attachmeni kernels

References



- Vary attachment kernel.
- Vary mechanisms:
 - Add edge deletion
 - 2. Add node deletion
 - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- ightharpoonup Q.: How does changing the model affect γ ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter?
- ► The answer is (surprisingly) yes.

Original model

Model details

Analysis

A more plausi mechanism

Redner & Krapivisky's model

Generalized mi Analysis Universality?

Sublinear attachment kernels

Superlinear attachment ernels

References



- Vary attachment kernel.
- Vary mechanisms:
 - Add edge deletion
 - Add node deletion
 - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- \triangleright Q.: How does changing the model affect γ ?
- Q.: Do we need preferential attachment and growth?
- Q: Do model details matter?
- ► The answer is (surprisingly) yes.

Original model

Analysis

Redner &

References



- Vary attachment kernel.
- Vary mechanisms:
 - Add edge deletion
 - Add node deletion
 - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter?
- ► The answer is (surprisingly) yes.

Original model

Model detail

A more plaus mechanism

Robustness
Redner &

Krapivisky's model

Analysis Jniversality? Sublinear attachme

kernels Superlinear attachmen

References



- Vary attachment kernel.
- Vary mechanisms:
 - Add edge deletion
 - Add node deletion
 - Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- \triangleright Q.: How does changing the model affect γ ?
- Q.: Do we need preferential attachment and growth?
- ► The answer is (surprisingly) yes.

Original model

Analysis

Redner &

References



- Vary attachment kernel.
- Vary mechanisms:
 - Add edge deletion
 - Add node deletion
 - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter?
- ► The answer is (surprisingly) yes.

Original model

Model details Analysis

mechanism Robustness

Redner & Krapivisky's model

nalysis Iniversality? Sublinear attachmen

Superlinear attac

.....

References



- Vary attachment kernel.
- Vary mechanisms:
 - Add edge deletion
 - Add node deletion
 - 3. Add edge rewiring
- Deal with directed versus undirected networks.
- Important Q.: Are there distinct universality classes for these networks?
- Q.: How does changing the model affect γ?
- Q.: Do we need preferential attachment and growth?
- Q.: Do model details matter?
- The answer is (surprisingly) yes.

Original model

Introduction Model details Analysis

A more plaus mechanism

Robustness
Redner &

edner & rapivisky's model

Analysis Jniversality? Sublinear attachme

Superlinear attac

References



Outline

Original model

Model details Analysis

A more plausible mechanism

Generalized model

Analysis

Superlinear attachment kernels

Original model

A more plausible mechanism

Redner &

References

Frame 23/57





- Let's look at preferential attachment (PA) a little more closely.
- ► PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ▶ For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- ▶ We need to know what everyone's degree is...
- ▶ PA is : an outrageous assumption of node capability.
- ▶ But a very simple mechanism saves the day...

Original model

Introduction
Model details
Analysis
A more plausible
mechanism

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment kernels
Superlinear attachmen

References



- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ▶ For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- ▶ We need to know what everyone's degree is...
- ▶ PA is : an outrageous assumption of node capability.
- ▶ But a very simple mechanism saves the day...

Original model

Introduction
Model details
Analysis
A more plausible
mechanism
Robustness

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment
kernels

References



- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ▶ For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- ▶ We need to know what everyone's degree is...
- ▶ PA is : an outrageous assumption of node capability.
- ▶ But a very simple mechanism saves the day...

Original model

Introduction
Model details
Analysis
A more plausible
mechanism

Redner & Krapivisky's model

Analysis Universality? Sublinear attachmen

Superlinear attachme kernels

References



- Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ▶ For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- ▶ We need to know what everyone's degree is...
- ▶ PA is : an outrageous assumption of node capability.
- ▶ But a very simple mechanism saves the day...

Original model Introduction Model details

Introduction
Model details
Analysis
A more plausible
mechanism
Robustness

Redner & Krapivisky's model

Generalized model Analysis Universality? Sublinear attachment

kernels

References



- ► Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ▶ For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- ▶ We need to know what everyone's degree is...
- ▶ PA is : an outrageous assumption of node capability.
- ▶ But a very simple mechanism saves the day...

Original model
Introduction
Model details

Analysis
A more plausible mechanism
Robustness

Redner & Krapivisky's model Generalized model Analysis

Analysis
Universality?
Sublinear attachment
kernels
Superlinear attachment

References

10101011000



- ▶ Let's look at preferential attachment (PA) a little more closely.
- PA implies arriving nodes have complete knowledge of the existing network's degree distribution.
- ▶ For example: If $P_{\text{attach}}(k) \propto k$, we need to determine the constant of proportionality.
- We need to know what everyone's degree is...
- ▶ PA is : an outrageous assumption of node capability.
- But a very simple mechanism saves the day...

Original model Introduction Model details Analysis A more plausible mechanism

Redner & Krapivisky's model Generalized model Analysis

Anaiysis Universality? Sublinear attachment kernels Superlinear attachmen

References



- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- ▶ We know that friends are weird...
- Assuming the existing network is random, we know probability of a random friend having degree k is

$$Q_k \propto kP_k$$

So rich-gets-richer scheme can now be seen to work in a natural way.

Original model

Model details
Analysis
A more plausible
mechanism
Robustness

Redner &
Krapivisky's model
Generalized model
Analysis

Universality?
Sublinear attachment kernels
Superlinear attachme

References



- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- ▶ We know that friends are weird...
- Assuming the existing network is random, we know probability of a random friend having degree k is

$$Q_k \propto k P_k$$

So rich-gets-richer scheme can now be seen to work in a natural way.

Original model

Introduction
Model details
Analysis
A more plausible
mechanism

Redner & Krapivisky's mode

Generalized model Analysis Universality? Sublinear attachmer

Superlinear attach

References



- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- We know that friends are weird...
- ▶ Assuming the existing network is random, we know probability of a random friend having degree *k* is

$Q_k \propto kP_k$

So rich-gets-richer scheme can now be seen to work in a natural way.

Original model

Introduction
Model details
Analysis
A more plausible
mechanism

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachmen

Superlinear attach

References



- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- We know that friends are weird...
- Assuming the existing network is random, we know probability of a random friend having degree k is

$Q_k \propto kP_k$

So rich-gets-richer scheme can now be seen to work in a natural way.

Original model

Introduction
Model details
Analysis
A more plausible
mechanism

Redner & Krapivisky's model

Generalized model Analysis Universality? Sublinear attachmen

Superlinear attach

References



- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- ▶ We know that friends are weird...
- Assuming the existing network is random, we know probability of a random friend having degree k is

$$Q_k \propto kP_k$$

So rich-gets-richer scheme can now be seen to work in a natural way.

Original model

Introduction
Model details
Analysis
A more plausible
mechanism

Redner & Krapivisky's model

Analysis
Universality?

kernels Superlinear attac

kernels

References



- Instead of attaching preferentially, allow new nodes to attach randomly.
- Now add an extra step: new nodes then connect to some of their friends' friends.
- Can also do this at random.
- ▶ We know that friends are weird...
- Assuming the existing network is random, we know probability of a random friend having degree k is

$$Q_k \propto kP_k$$

So rich-gets-richer scheme can now be seen to work in a natural way.

Original model

Model details
Analysis
A more plausible
mechanism
Robustness

Redner & Krapivisky's model

Generalized model Analysis Universality? Sublinear attachme

Superlinear atta

References



Outline

Original model

Model details

Analysis

A more plausible mechanism

Robustness

Generalized model

Analysis

Superlinear attachment kernels

Original model

Robustness

Redner &

References

Frame 26/57



Robustness

- We've looked at some aspects of contagion on scale-free networks:
- Another simple story concerns system robustness.
- ► Albert et al., Nature, 2000:

Original model

Robustness

Redner &

References



scale-free networks:

► Albert et al., Nature, 2000:

We've looked at some aspects of contagion on

Another simple story concerns system robustness.

Facilitate disease-like spreading.

References



scale-free networks:

► Albert et al., Nature, 2000:

We've looked at some aspects of contagion on

Another simple story concerns system robustness.

 Facilitate disease-like spreading. Inhibit threshold-like spreading.



scale-free networks:

► Albert et al., Nature, 2000:

We've looked at some aspects of contagion on

Another simple story concerns system robustness.

 Facilitate disease-like spreading. Inhibit threshold-like spreading.



References

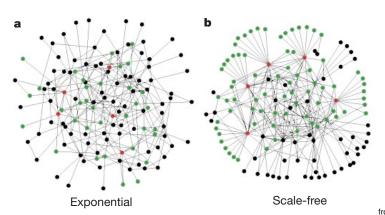




- We've looked at some aspects of contagion on scale-free networks:
 - Facilitate disease-like spreading.
 - Inhibit threshold-like spreading.
- Another simple story concerns system robustness.
- Albert et al., Nature, 2000:
 - "Error and attack tolerance of complex networks" [1]

Robustness

 Standard random networks (Erdős-Rényi) versus
 Scale-free networks



Original model

Introduction Model details Analysis A more plausible

Robustness

Redner & Krapivisky's model

Generalized model Analysis Universality? Sublinear attachment kernels

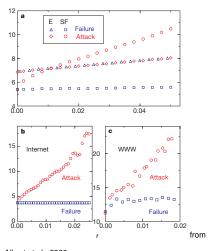
References

from

Frame 28/57



Robustness



 Plots of network diameter as a function of fraction of nodes removed

- Erdős-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

Original model

Model details
Analysis
A more plausible
mechanism
Robustness

Redner & Krapivisky's model

Generalized model
Analysis
Universality?

Superlinear attach kernels

References

References

Frame 29/57



Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- ▶ All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
 - Physically larger nodes that may be harder to 'target'
 or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

Original model
Introduction

Model details Analysis A more plausibl

Robustness
Redner &

Redner &
Krapivisky's model
Generalized model

anaiysis Jniversality? Sublinear attachn

subilitear attacilit kernels Superlinear attach

uperlinear attachi ernels

References



- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
 - Physically larger nodes that may be harder to 'target'
 or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

Original model
Introduction

Model details
Analysis

Robustness
Redner &

Krapivisky's model

Generalized model

Analysis

Universality?

Sublinear attachment kernels Superlinear attachmen

erreis

References



- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- ▶ But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
 - Physically larger nodes that may be harder to 'target'
 or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

Original model
Introduction

Introduction
Model details
Analysis

Robustness

Redner & Krapivisky's model

Generalized model Analysis Universality?

Sublinear attachment ernels Superlinear attachmer

References



Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- ▶ But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
 - Physically larger nodes that may be harder to 'target'
 or subnetworks of smaller, normal-sized nodes.
- ▶ Need to explore cost of various targeting schemes.

Original model
Introduction
Model details
Analysis
A more plausible
mechanism
Robustness
Redner &
Krapivisky's mod
Generalized model
Analysis

kerneis

References



Robustness

- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- ▶ But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
 - 1. Physically larger nodes that may be harder to 'target'
 - 2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

Original model
Introduction
Model details
Analysis
A more plausible
mechanism
Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attach

kernels Superlinear attach

Superlinear attach kernels

References



- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
 - 1. Physically larger nodes that may be harder to 'target'
 - 2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

Original model

Model details Analysis A more plausit mechanism Bobustness

Redner & Krapivisky's model

Analysis

Universality?

Sublinear attachment

Superlinear attach kernels

References



- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
 - 1. Physically larger nodes that may be harder to 'target'
 - 2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

Original model **Robustness** Redner &

References



- Scale-free networks are thus robust to random failures yet fragile to targeted ones.
- All very reasonable: Hubs are a big deal.
- ▶ But: next issue is whether hubs are vulnerable or not.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
 - 1. Physically larger nodes that may be harder to 'target'
 - 2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

Original model
Introduction
Model details

Analysis
A more plausible
mechanism

Robustness

Redner & Krapivisky's model

Universality?
Sublinear attachment
kernels
Superlinear attachme

ernels

References



Outline

Model details

Analysis

A more plausible mechanism

Redner & Krapivisky's model

Generalized model

Analysis

Superlinear attachment kernels

Original model

Redner &

Generalized model

References

Frame 31/57





Fooling with the mechanism:

▶ 2001: Redner & Krapivsky (RK) [3] explored the general attachment kernel:

Pr(attach to node
$$i$$
) $\propto A_k = k_i^{\nu}$

where A_k is the attachment kernel and $\nu > 0$.

- RK also looked at changing the details of the attachment kernel.
- ▶ We'll follow RK's approach using rate equations (⊞).

Original model

Model details
Analysis
A more plausible

Redner & Kranivisky's mode

Generalized model
Analysis
Universality?

kernels Superlinear attachn

kernels

References



Fooling with the mechanism:

▶ 2001: Redner & Krapivsky (RK) [3] explored the general attachment kernel:

Pr(attach to node
$$i$$
) $\propto A_k = k_i^{\nu}$

where A_k is the attachment kernel and $\nu > 0$.

- RK also looked at changing the details of the attachment kernel.
- ▶ We'll follow RK's approach using rate equations (⊞).

Original model

Model details
Analysis
A more plausible

Redner & Krapivisky's mode

Generalized model
Analysis
Universality?
Sublinear attachment

kernels

References



Fooling with the mechanism:

▶ 2001: Redner & Krapivsky (RK) [3] explored the general attachment kernel:

Pr(attach to node
$$i$$
) $\propto A_k = k_i^{\nu}$

where A_k is the attachment kernel and $\nu > 0$.

- RK also looked at changing the details of the attachment kernel.
- ▶ We'll follow RK's approach using rate equations (⊞).

Original model

Model details

Analysis

A more plausible

Redner & Krapivisky's mode

Generalized model
Analysis
Universality?
Sublinear attachment

Superlinear attach kernels

References



Fooling with the mechanism:

▶ 2001: Redner & Krapivsky (RK) [3] explored the general attachment kernel:

Pr(attach to node
$$i$$
) $\propto A_k = k_i^{\nu}$

where A_k is the attachment kernel and $\nu > 0$.

- RK also looked at changing the details of the attachment kernel.
- ▶ We'll follow RK's approach using rate equations (\boxplus).

Original model

Model details Analysis A more plausible

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

kernels

References



$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- The second term corresponds to degree k nodes becoming degree k – 1 nodes.
- 3. Detail: $A_0 = 0$
- 4. One node is added per unit time.
- 5. Seed with some initial network (e.g., a connected pair)

Original model

Introduction Model details

Analysis A more plausib

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

erneis

References



$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- The second term corresponds to degree k nodes becoming degree k – 1 nodes.
- 3. Detail: $A_0 = 0$
- 4. One node is added per unit time.
- 5. Seed with some initial network (e.g., a connected pair)

Original model

Model detail

Analysis
A more plausib

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

Superlinear attachmer ternels

References



$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 2. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 3. Detail: $A_0 = 0$
- 4. One node is added per unit time.
- 5. Seed with some initial network (e.g., a connected pair)

Original model

Introduction Model detai

Analysis A more plausit

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

ernels

References



$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 2. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 3. Detail: $A_0 = 0$
- 4. One node is added per unit time.
- 5. Seed with some initial network (e.g., a connected pair)

Original model

Introduction Model detai

Analysis A more plausib

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

ernels

References



$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 2. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 3. Detail: $A_0 = 0$
- 4. One node is added per unit time.
- 5. Seed with some initial network (e.g., a connected pair)

Original model

Introduction Model detail

Analysis A more plausib

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

References



$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 2. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 3. Detail: $A_0 = 0$
- 4. One node is added per unit time.
- Seed with some initial network (e.g., a connected pair)

Original model

Introduction Model detail

Analysis
A more plausit

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

kerneis

References



$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

where N_k is the number of nodes of degree k.

- 1. The first term corresponds to degree k-1 nodes becoming degree k nodes.
- 2. The second term corresponds to degree k nodes becoming degree k-1 nodes.
- 3. Detail: $A_0 = 0$
- 4. One node is added per unit time.
- 5. Seed with some initial network (e.g., a connected pair)

Original model

Introduction Model detai

Analysis
A more plausit

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

References

ricicionoco



Outline

Model details

Analysis

A more plausible mechanism

Redner & Krapivisky's model

Analysis

Superlinear attachment kernels

Original model

Redner &

Analysis

References

Frame 34/57





In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} kN_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

▶ Detail: we are ignoring initial seed network's edges.

Original model

Model details Analysis

nechanism Robustness

Redner & Krapivisky's mode

Analysis

Universality? Sublinear attachm

Superlinear attachm

References



In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} kN_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

▶ Detail: we are ignoring initial seed network's edges.

Original model

Model details Analysis A more plausibl

Redner & Krapivisky's mode

Analysis
Universality?

Sublinear attachment kernels Superlinear attachment

References



In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} kN_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

▶ Detail: we are ignoring initial seed network's edges.

Original model
Introduction

Model details Analysis

nechanism Robustness

Redner & Krapivisky's model

Analysis Universality?

Sublinear attachment kernels Superlinear attachment

References



In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} kN_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

▶ Detail: we are ignoring initial seed network's edges.

Original model
Introduction

Model details

Analysis

A more plausit nechanism Robustness

Redner & Krapivisky's model

Analysis

Universality

Sublinear attachment kernels

Superlinear attach kernels

References



In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} kN_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

Detail: we are ignoring initial seed network's edges.

Original model

Analysis
A more plausible

Redner & Krapivisky's model

Analysis
Universality?

Sublinear attachment kernels Superlinear attachment

References



In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} kN_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

Detail: we are ignoring initial seed network's edges.

Original model
Introduction

Model details Analysis A more plausibl

Redner & Krapivisky's model

Analysis Universality?

kernels Superlinear attachment

References



In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} kN_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

Detail: we are ignoring initial seed network's edges.

Original model
Introduction

Model details Analysis A more plausibl

Redner & Krapivisky's model

Analysis Universality?

kernels Superlinear attachment

References



In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} kN_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

Detail: we are ignoring initial seed network's edges.

Original model
Introduction

Model details Analysis A more plausibl

Redner & Krapivisky's model

Analysis

Jniversality? Sublinear attachmen

Superlinear attachment ernels

References



In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} kN_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

Detail: we are ignoring initial seed network's edges.

Original model

Model details Analysis A more plausible

Redner & Krapivisky's model

Analysis Universality?

kernels Superlinear attachment

References



In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} kN_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

Detail: we are ignoring initial seed network's edges.

Original model
Introduction

Model details Analysis A more plausible

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

ernels Superlinear attachment ernels

References



In general, probability of attaching to a specific node of degree k at time t is

$$\mathbf{Pr}(\text{attach to node } i) = \frac{A_k}{A(t)}$$

where
$$A(t) = \sum_{k=1}^{\infty} A_k N_k(t)$$
.

- ▶ E.g., for BA model, $A_k = k$ and $A = \sum_{k=1}^{\infty} kN_k(t)$.
- ▶ For $A_k = k$, we have

$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) = 2t$$

since one edge is being added per unit time.

▶ Detail: we are ignoring initial seed network's edges.

Original model
Introduction
Model details

Model details Analysis A more plausible nechanism

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

ernels Superlinear attachment ernels

References



$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t} \left[(k-1)N_{k-1} - kN_k \right] + \delta_{k1}$$

- ► As for BA method, look for steady-state growing solution: N_k = n_kt.
- ▶ We replace dN_k/dt with $dn_kt/dt = n_k$.
- ▶ We arrive at a difference equation:

$$n_k = \frac{1}{2t} [(k-1)n_{k-1}t - kn_kt] + \delta_{k1}$$

Original model

Model detail:

Analysis A more plausit

Redner & Krapivisky's model

Generalized model Analysis

Universality? Sublinear attachme

Superlinear attachment

References



$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t} \left[(k-1)N_{k-1} - kN_k \right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution: N_k = n_kt.
- ▶ We replace dN_k/dt with $dn_kt/dt = n_k$.
- ▶ We arrive at a difference equation:

$$n_k = \frac{1}{2t} [(k-1)n_{k-1}t - kn_kt] + \delta_{k1}$$

Original model

Model detail

Analysis A more plausib

Redner & Kranivisky's model

Generalized model

Analysis Universality?

Sublinear attachment kernels

References



$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t} \left[(k-1)N_{k-1} - kN_k \right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution: $N_k = n_k t$.
- ▶ We replace dN_k/dt with $dn_kt/dt = n_k$.
- ▶ We arrive at a difference equation:

$$n_k = \frac{1}{2t}[(k-1)n_{k-1}t - kn_kt] + \delta_{k1}$$

Original model

Model detail:

Analysis A more plausib

Robustness
Redner &

rapivisky's model

Analysis

Universality? Sublinear attachment kernels

Superlinear attachment ernels

References



$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t} \left[(k-1)N_{k-1} - kN_k \right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution: $N_k = n_k t$.
- ▶ We replace dN_k/dt with $dn_kt/dt = n_k$.
- ▶ We arrive at a difference equation:

$$n_k = \frac{1}{2t} [(k-1)n_{k-1}t - kn_kt] + \delta_{k1}$$

Original model

Model detail

Analysis A more plausib

Redner & Krapivisky's model

Generalized me Analysis

Universality? Sublinear attachme

Superlinear attachmen kernels

References



So now

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{A} \left[A_{k-1} N_{k-1} - A_k N_k \right] + \delta_{k1}$$

becomes

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = \frac{1}{2t} \left[(k-1)N_{k-1} - kN_k \right] + \delta_{k1}$$

- As for BA method, look for steady-state growing solution: N_k = n_kt.
- ▶ We replace dN_k/dt with $dn_kt/dt = n_k$.
- ▶ We arrive at a difference equation:

$$n_k = \frac{1}{2!} [(k-1)n_{k-1}! - kn_k!] + \delta_{k1}$$

Original model

Introduction Model detail

Analysis A more plausit

Redner &

Crapivisky s model

Analysis

Universality?
Sublinear attachment

Superlinear attachmen kernels

References

Frame 36/57



Rearrange and simply:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

$$\Rightarrow (k+2)n_k = (k-1)n_{k-1} + 2\delta_k$$

► Two cases:

$$k = 1 : n_1 = 2/3 \text{ since } n_0 = 0$$

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1}$$

Original model

Redner &

Analysis

References



$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

$$\Rightarrow (k+2)n_k = (k-1)n_{k-1} + 2\delta_{k1}$$

► Two cases:

 $k = 1 : n_1 = 2/3$ since $n_0 = 0$

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1}$$

Original model

Introduction Model detail

Analysis A more plausi

mechanism Robustness

Redner & Krapivisky's mode

Generalized model

Analysis

Universality? Sublinear attachme

Superlinear attachm

References

reletetices



Rearrange and simply:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

$$\Rightarrow (k+2)n_k = (k-1)n_{k-1} + 2\delta_{k1}$$

▶ Two cases:

$$k = 1 : n_1 = 2/3$$
 since $n_0 = 0$

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1}$$

Original model

Model detail

Analysis A more plausi

nechanism Robustness

Redner & Krapivisky's mode

Generalized model
Analysis

Universality?

Sublinear attachment kernels

superiinear attachme kernels

References



Rearrange and simply:

$$n_k = \frac{1}{2}(k-1)n_{k-1} - \frac{1}{2}kn_k + \delta_{k1}$$

$$\Rightarrow (k+2)n_k = (k-1)n_{k-1} + 2\delta_{k1}$$

Two cases:

$$k = 1 : n_1 = 2/3$$
 since $n_0 = 0$

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1}$$

Original model

Model detail

A more plausit

Redner & Krapivisky's mode

Generalized model

Analysis Universality?

Sublinear attachment kernels

References



Now find n_k :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} \frac{(k-4)}{k-1} n_{k-4}$$

$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)} n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$$

Original model

Model details

more plausible lechanism

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

kernels Superlinear attachment

References



Original model

Redner &

Analysis

Now find n_k :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k+1} \frac{(k-4)}{k+1} n_{k-4}$$

kerneis

References

 $\Rightarrow n_k = \frac{6}{k(k+1)(k+2)} n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$



Now find n_k :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k+1} \frac{(k-4)}{k+1} n_{k-4}$$

References

Original model

Redner &

Analysis

 $\Rightarrow n_k = \frac{6}{k(k+1)(k+2)} n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$



Original model

Now find n_k :

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} \frac{(k-4)}{k-1} n_{k-4}$$

Redner &

Analysis

References

 $\Rightarrow n_k = \frac{6}{k(k+1)(k+2)} n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$



Now find n_k :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} \frac{(k-4)}{k-1} n_{k-4}$$

$$=\frac{(k-1)}{k+2}\frac{(k-2)}{k+1}\frac{(k-3)}{k}\frac{(k-4)}{(k-1)}\frac{(k-5)}{(k-2)}\cdots\frac{5}{8}\frac{4}{7}\frac{3}{6}\frac{2}{5}\frac{1}{4}n_1$$

$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)} n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$$

Original model

Model details
Analysis

more plausible nechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachme

Sublinear attachment kernels Superlinear attachment

References



Now find n_k :

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} \frac{(k-4)}{k-1} n_{k-4}$$

Model details
Analysis
A more plausible mechanism

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachmen

kernels Superlinear attachmen kernels

References

$$=\frac{(k-1)}{k+2}\frac{(k-2)}{k+1}\frac{(k-3)}{k}\frac{(k-4)}{(k-1)}\frac{(k-5)}{(k-2)}\cdots\frac{5}{8}\frac{\cancel{4}}{7}\frac{3}{6}\frac{2}{5}\frac{1}{\cancel{4}}n_1$$

$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)}n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$$



Original model

Generalized model

Now find n_k :

$$k > 1: n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} \frac{(k-4)}{k-1} n_{k-4}$$

Analysis
Universality?

Redner &

Sublinear attachment kernels

References

$$=\frac{(k-1)}{k+2}\frac{(k-2)}{k+1}\frac{(k-3)}{k}\frac{(k-4)}{(k-1)}\frac{(k-5)}{(k-2)}\cdots\frac{5}{8}\frac{4}{7}\frac{3}{6}\frac{2}{5}\frac{1}{4}n_1$$

$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)}n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$$



Now find n_k :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} \frac{(k-4)}{k-1} n_{k-4}$$

Original model

Introduction

Model details Analysis A more plausible

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

Superlinear attachmer ernels

References

$$=\frac{(k-1)}{k+2}\frac{(k-2)}{k+1}\frac{(k-3)}{k}\frac{(k-4)}{(k-1)}\frac{(k-5)}{(k-2)}\cdots\frac{5}{8}\frac{4}{7}\frac{3}{5}\frac{2}{4}n_{1}$$

$$\Rightarrow n_{k} = \frac{6}{k(k+1)(k+2)} n_{1} = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$$



Now find n_k :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} \frac{(k-4)}{k-1} n_{k-4}$$

References

Model details Analysis A more plausible mechanism

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

Superlinear attachme kernels

$$= \frac{(k-1)(k-2)(k-3)(k-4)(k-5)\cdots \cancel{5}\cancel{4}\cancel{3}\cancel{2}\cancel{1}\cancel{1}\cancel{1}\cancel{1}\cancel{2}\cancel{1}\cancel{1}\cancel{2}\cancel{1}\cancel{2}\cancel{2}\cancel{2}\cancel{2}\cancel{3}\cancel{3}\cancel{3}\cancel{3}\cancel{3}\cancel{4}\cancel{1}\cancel{1}}{(k-1)(k-2)}$$

$$\Rightarrow n_{k} = \frac{6}{k(k+1)(k+2)}n_{1} = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$$



Now find n_k :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} \frac{(k-4)}{k-1} n_{k-4}$$

$$= \frac{(k-1)(k-2)(k-3)(k-4)(k-5)\cdots \cancel{5}\cancel{4}\cancel{3}\cancel{2}\cancel{1}}{(k-1)(k-2)\cdots \cancel{8}\cancel{7}\cancel{8}\cancel{5}\cancel{4}\cancel{7}$$

$$\Rightarrow \frac{n_k}{k(k+1)(k+2)}n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$$

Original model

Model details Analysis

Redner &

Krapivisky's model Generalized model Analysis

Universality?
Sublinear attachment kernels
Superlinear attachment

References



Now find n_k :

$$k > 1 : n_k = \frac{(k-1)}{k+2} n_{k-1} = \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} n_{k-2}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} n_{k-3}$$
$$= \frac{(k-1)}{k+2} \frac{(k-2)}{k+1} \frac{(k-3)}{k} \frac{(k-4)}{k-1} n_{k-4}$$

$$= \frac{(k-1)(k-2)(k-3)(k-4)(k-5)\cdots 5\cancel{4}\cancel{3}\cancel{2}\cancel{1}}{(k-1)(k-2)\cdots \cancel{8}\cancel{7}\cancel{8}\cancel{5}\cancel{4}\cancel{7}\cancel{1}} n_1$$

$$\Rightarrow n_k = \frac{6}{k(k+1)(k+2)}n_1 = \frac{4}{k(k+1)(k+2)} \sim k^{-3}$$

Original model

Model details
Analysis

Redner &

Krapivisky's model Generalized model Analysis

Universality?
Sublinear attachment kernels
Superlinear attachment

References



Outline

Model details

Analysis

A more plausible mechanism

Redner & Krapivisky's model

Generalized model

Analysis

Universality?

Superlinear attachment kernels

Original model

Redner &

Universality?

References

Frame 39/57





Universality?

As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}$$
 for large k .

- Now: what happens if we start playing around with the attachment kernel A_k ?
- ▶ Again, is the result $\gamma = 3$ universal (⊞)?
- ▶ Natural modification: $A_k = k^{\nu}$ with $\nu \neq 1$.
- ▶ But we'll first explore a more subtle modification of A_k made by Redner/Krapivsky [3]
- ► Keep A_k linear in k but tweak details.
- ▶ Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

Original model

ntroduction Model details Analysis A more plausible nechanism

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment
kernels
Superlinear attachment

References



Universality?

As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}$$
 for large k .

- Now: what happens if we start playing around with the attachment kernel A_k?
- ▶ Again, is the result $\gamma = 3$ universal (⊞)?
- ▶ Natural modification: $A_k = k^{\nu}$ with $\nu \neq 1$.
- ▶ But we'll first explore a more subtle modification of A_k made by Redner/Krapivsky [3]
- ▶ Keep A_k linear in k but tweak details.
- ▶ Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

Original model

froduction flodel details unalysis umore plausible nechanism

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment
kernels
Superlinear attachment

References



Universality?

As expected, we have the same result as for the BA model:

$$N_k(t) = n_k(t)t \propto k^{-3}$$
 for large k .

- Now: what happens if we start playing around with the attachment kernel A_k?
- ▶ Again, is the result $\gamma = 3$ universal (⊞)?
- ▶ Natural modification: $A_k = k^{\nu}$ with $\nu \neq 1$.
- ▶ But we'll first explore a more subtle modification of A_k made by Redner/Krapivsky [3]
- ► Keep A_k linear in k but tweak details.
- ▶ Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

Original model

Model details Analysis A more plausible nechanism

Redner & Krapivisky's model

Krapivisky's mode Generalized model Analysis Universality? Sublinear attachment kernels Superlinear attachment

References



$$N_k(t) = n_k(t)t \propto k^{-3}$$
 for large k .

- Now: what happens if we start playing around with the attachment kernel A_k ?
- ▶ Again, is the result $\gamma = 3$ universal (\boxplus)?
- ▶ Natural modification: $A_k = k^{\nu}$ with $\nu \neq 1$.
- ▶ But we'll first explore a more subtle modification of A_K made by Redner/Krapivsky [3]
- ▶ Keep A_k linear in k but tweak details.
- ▶ Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

Original model

ntroduction Model details Analysis A more plausible nechanism

Redner & Krapivisky's model

Krapivisky's model
Generalized model
Analysis
Universality?
Sublinear attachment
kernels
Superlinear attachment

References



$$N_k(t) = n_k(t)t \propto k^{-3}$$
 for large k .

- Now: what happens if we start playing around with the attachment kernel A_k?
- ▶ Again, is the result $\gamma = 3$ universal (\boxplus)?
- ▶ Natural modification: $A_k = k^{\nu}$ with $\nu \neq 1$.
- ▶ But we'll first explore a more subtle modification of A_k made by Redner/Krapivsky [3]
- ► Keep A_k linear in k but tweak details.
- ▶ Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

Original model

ntroduction

Model details

Analysis

A more plausible
nechanism

Redner & Krapivisky's mode

Krapivisky's model
Generalized model
Analysis
Universality?
Sublinear attachment
kernels
Superlinear attachment

References



$$N_k(t) = n_k(t)t \propto k^{-3}$$
 for large k .

- Now: what happens if we start playing around with the attachment kernel A_k?
- ▶ Again, is the result $\gamma = 3$ universal (\boxplus)?
- ▶ Natural modification: $A_k = k^{\nu}$ with $\nu \neq 1$.
- ▶ But we'll first explore a more subtle modification of A_k made by Redner/Krapivsky [3]
- ▶ Keep A_k linear in k but tweak details.
- ▶ Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

Original model

ntroduction Model details Analysis A more plausible nechanism

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment
kernels
Superlinear attachment

References



$$N_k(t) = n_k(t)t \propto k^{-3}$$
 for large k .

- Now: what happens if we start playing around with the attachment kernel A_k?
- ▶ Again, is the result $\gamma = 3$ universal (\boxplus)?
- ▶ Natural modification: $A_k = k^{\nu}$ with $\nu \neq 1$.
- ▶ But we'll first explore a more subtle modification of A_k made by Redner/Krapivsky^[3]
- ▶ Keep A_k linear in k but tweak details.
- ▶ Idea: Relax from $A_k = k$ to $A_k \sim k$ as $k \to \infty$.

Original model

Model details
Analysis
A more plausible
mechanism

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment
kernels
Superlinear attachment

References



$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t$$
 for large t .

We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

- ▶ We assume that $A = \mu t$
- We'll find μ later and make sure that our assumption is consistent.
- ▶ As before, also assume $N_k(t) = n_k t$.

Original model

Model details

Analysis A more plausil

Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment
kernels

erneis

References



$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t$$
 for large t .

We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

- \blacktriangleright We assume that $A = \mu t$
- We'll find μ later and make sure that our assumption is consistent.
- ▶ As before, also assume $N_k(t) = n_k t$.

Original model

Model details

Analysis A more plaus

mechanism Robustness

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment

Superlinear attachm ernels

References



$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t$$
 for large t .

We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

- \blacktriangleright We assume that $A = \mu t$
- \triangleright We'll find μ later and make sure that our assumption
- ▶ As before, also assume $N_k(t) = n_k t$.

Original model

Redner &

Universality?

References



$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t$$
 for large t .

We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

- ▶ We assume that $A = \mu t$
- We'll find μ later and make sure that our assumption is consistent.
- As before, also assume $N_k(t) = n_k t$.

Original model

Model detai

A more plausi mechanism

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment kernels

References

relefences



$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t$$
 for large t .

We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

- We assume that $A = \mu t$
- ▶ We'll find μ later and make sure that our assumption is consistent.
- As before, also assume $N_k(t) = n_k t$.

Original model

Model detail

A more plausi mechanism

Redner &

Generalized model
Analysis
Universality?
Sublinear attachment
kernels

References



$$A(t) = \sum_{k'=1}^{\infty} k' N_{k'}(t) \simeq 2t$$
 for large t .

We now have

$$A(t) = \sum_{k'=1}^{\infty} A_{k'} N_{k'}(t)$$

where we only know the asymptotic behavior of A_k .

- We assume that $A = \mu t$
- We'll find µ later and make sure that our assumption is consistent.
- ▶ As before, also assume $N_k(t) = n_k t$.

Original model

Redner &

Universality?

References



▶ For $A_k = k$ we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

▶ This now becomes

$$n_k = \frac{1}{\mu} \left[A_{k-1} n_{k-1} - A_k n_k \right] + \delta_k$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu \delta_{k1}$$

Again two cases:

$$k = 1: n_1 = \frac{\mu}{\mu + A_1}.$$

$$k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

Original model

Introduction

Model deta

A more plaus mechanism

Robustness

Redner & Krapivisky's model

Analysis
Universality?

Universality? Sublinear attac

Sublinear attachment kernels

Reffiels

References

Frame 42/57



ightharpoonup For $A_k = k$ we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

▶ This now becomes

$$n_k = \frac{1}{\mu} [A_{k-1} n_{k-1} - A_k n_k] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_k$$

Again two cases:

$$k = 1: n_1 = \frac{\mu}{\mu + A_1}$$

$$k > 1: n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

Original model

Introduction

Model detai

A more plausit nechanism

Redner & Krapivisky's model

Generalized m Analysis Universality?

Universality? Sublinear attack

kernels

Kerriers

References

Frame 42/57



ightharpoonup For $A_k = k$ we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

► This now becomes

$$n_k = \frac{1}{\mu} \left[A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

Again two cases:

$$k = 1: n_1 = \frac{\mu}{\mu + A_1}$$

$$k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

Original model

Introduction

Analysis

A more plausib mechanism Robustness

Redner & Krapivisky's model

Generalized mod Analysis Universality?

Universality? Sublinear attack

Sublinear attachment kernels

Superlinear attachmer kernels

References



▶ For $A_k = k$ we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

► This now becomes

$$n_k = \frac{1}{\mu} \left[A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

$$\Rightarrow (A_k + \mu)n_k = A_{k-1}n_{k-1} + \mu\delta_{k1}$$

Again two cases:

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1}.$$

$$k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

Original model

Model deta

Analysis A more plausit

Redner &

Analysis
Universality?
Sublinear attachment

Superlinear attachmer kernels

References

Frame 42/57



ightharpoonup For $A_k = k$ we had

$$n_k = \frac{1}{2} [(k-1)n_{k-1} - kn_k] + \delta_{k1}$$

► This now becomes

$$n_k = \frac{1}{\mu} \left[A_{k-1} n_{k-1} - A_k n_k \right] + \delta_{k1}$$

$$\Rightarrow (\mathbf{A}_k + \mu)\mathbf{n}_k = \mathbf{A}_{k-1}\mathbf{n}_{k-1} + \mu \delta_{k1}$$

Again two cases:

$$k = 1 : n_1 = \frac{\mu}{\mu + A_1}.$$

$$k > 1 : n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k}$$

Original model

Model deta

Analysis A more plausit

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

Superlinear attachmi

References

Frame 42/57



Dealing with the k > 1 case:

$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= n_1 \frac{A_1}{A_k} \left(1 + \frac{\mu}{A_1} \right) \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= \frac{\mu}{A_k} \prod_{i=1}^k \frac{1}{1 + \frac{\mu}{A_i}} \text{ since } n_1 = \mu/(\mu + A_1)$$

Original model

Model details Analysis

A more plausible mechanism Robustness

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment kernels

n /

References

Frame 43/57



$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= n_1 \frac{A_1}{A_k} \left(1 + \frac{\mu}{A_1} \right) \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= \frac{\mu}{A_k} \prod_{i=1}^k \frac{1}{1 + \frac{\mu}{A_i}} \text{ since } n_1 = \mu/(\mu + A_1)$$

Original model

Model details

Analysis
A more plausibl

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment

Superlinear attachme kernels

References



$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_{k-2} \frac{A_{k-2}}{A_{k-1}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= n_1 \frac{A_1}{A_k} \left(1 + \frac{\mu}{A_1} \right) \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} \text{ since } n_1 = \mu/(\mu + A_1)$$

Original model

Introduction

Analysis

Robustness
Redner &

Generalized model
Analysis

Universality? Sublinear attachment kernels

Poforonoos

References



$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_{k-2} \frac{A_{k-2}}{A_{k_1}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= n_1 \frac{A_1}{A_k} \left(1 + \frac{\mu}{A_1} \right) \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= \frac{\mu}{A_k} \prod_{i=1}^k \frac{1}{1 + \frac{\mu}{A_i}} \text{ since } n_1 = \mu/(\mu + A_1)$$

Original model

Redner &

Universality?

References



$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_{k-2} \frac{A_{k-2}}{A_{k_1}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$
$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= n_1 \frac{A_1}{A_k} \left(1 + \frac{\mu}{A_1} \right) \prod_{j=1}^{n} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= \frac{\mu}{A_k} \prod_{i=1}^k \frac{1}{1 + \frac{\mu}{A_i}}$$
 since $n_1 = \mu/(\mu + A_1)$

Original model

Introduction Model detai

Analysis A more plausib

Robustness
Redner &

Krapivisky's model

Generalized model

Universality? Sublinear attachmen

Superlinear attachment kernels

References



$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_{k-2} \frac{A_{k-2}}{A_{k_1}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= n_1 \frac{A_1}{A_k} \left(1 + \frac{\mu}{A_1} \right) \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$=\frac{\mu}{A_k}\prod_{i=1}^k\frac{1}{1+\frac{\mu}{A_i}}$$
 since $n_1=\mu/(\mu+A_1)$

Original model

Introduction Model detai

Analysis A more plausib

Redner & Krapivisky's model

Generalized model
Analysis
Universality?

Sublinear attachment kernels

References

References



$$n_k = n_{k-1} \frac{A_{k-1}}{\mu + A_k} = n_{k-1} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_{k-2} \frac{A_{k-2}}{A_{k_1}} \frac{1}{1 + \frac{\mu}{A_{k-1}}} \frac{A_{k-1}}{A_k} \frac{1}{1 + \frac{\mu}{A_k}}$$

$$= n_1 \frac{A_1}{A_k} \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= n_1 \frac{A_1}{A_k} \left(1 + \frac{\mu}{A_1} \right) \prod_{j=2}^k \frac{1}{1 + \frac{\mu}{A_j}}$$

$$= \frac{\mu}{A_k} \prod_{i=1}^k \frac{1}{1 + \frac{\mu}{A_i}}$$
 since $n_1 = \mu/(\mu + A_1)$

Original model

Model details Analysis

Analysis A more plausible nechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachme

kernels Superlinear attachment kernels

References



- ▶ Time for pure excitement: Find asymptotic behavior of n_k given $A_k \to k$ as $k \to \infty$.
- ► For large *k*:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} = \frac{\mu}{A_k} \prod_{j=1}^k \frac{A_j}{A_j + \mu}$$

 $= \frac{\mu}{A_{k}} \frac{A_{1}}{(A_{1} + \mu)} \frac{A_{2}}{(A_{2} + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{K}{(k+\mu)}$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi}k^{k+1/2}e^{-k}}{\sqrt{2\pi}(k+\mu+1)^{k+\mu+1+1/2}e^{-(k+\mu+1)}}$$

 $\propto k^{-\mu-1}$

▶ Since μ depends on A_k , details matter...

Original model

Model details

Analysis A more plausibl

nechanism Robustness

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment

Superlinear attachr

References



- ▶ Time for pure excitement: Find asymptotic behavior of n_k given $A_k \to k$ as $k \to \infty$.
- ► For large *k*:

$$n_{k} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{A_{j}}{A_{j} + \mu}$$

$$= \frac{\mu}{A_{k}} \frac{A_{1}}{(A_{1} + \mu)} \frac{A_{2}}{(A_{2} + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{k}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi}k^{k+1/2}e^{-k}}{\sqrt{2\pi}(k+\mu+1)^{k+\mu+1+1/2}e^{-(k+\mu+1)}}$$

$$\propto k^{-\mu-1}$$

▶ Since μ depends on A_k , details matter...

Original model

Model details Analysis

A more plausible nechanism

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

kernels Superlinear attachment kernels

References



- ▶ Time for pure excitement: Find asymptotic behavior of n_k given $A_k \to k$ as $k \to \infty$.
- ► For large *k*:

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \frac{1}{1 + \frac{\mu}{A_j}} = \frac{\mu}{A_k} \prod_{j=1}^k \frac{A_j}{A_j + \mu}$$

$$=\frac{\mu}{\cancel{A_{K}}}\frac{A_{1}}{(A_{1}+\mu)}\frac{A_{2}}{(A_{2}+\mu)}\cdots\frac{k-1}{(k-1+\mu)}\frac{\cancel{K}}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi}k^{k+1/2}e^{-k}}{\sqrt{2\pi}(k+\mu+1)^{k+\mu+1+1/2}e^{-(k+\mu+1)}}$$

$$\propto k^{-\mu-1}$$

▶ Since μ depends on A_k , details matter...

Original model

Model details

Analysis A more plausil

nechanism Robustness

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachmen

kernels
Superlinear attachment

References



Redner &

Universality?

References

Universality?

- ▶ Time for pure excitement: Find asymptotic behavior of n_k given $A_k \to k$ as $k \to \infty$.
- ► For large *k*:

$$n_{k} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{A_{j}}{A_{j} + \mu}$$

$$= \frac{\mu}{A_{k}} \frac{A_{1}}{(A_{1} + \mu)} \frac{A_{2}}{(A_{2} + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{k}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi}k^{k+1/2}e^{-k}}{\sqrt{2\pi}(k+\mu+1)^{k+\mu+1+1/2}e^{-(k+\mu+1)}}$$

 $\propto k^{-\mu-1}$

▶ Since μ depends on A_k , details matter...



- ▶ Time for pure excitement: Find asymptotic behavior of n_k given $A_k \to k$ as $k \to \infty$.
- ► For large *k*:

$$n_{k} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{A_{j}}{A_{j} + \mu}$$

$$= \frac{\mu}{A_{k}} \frac{A_{1}}{(A_{1} + \mu)} \frac{A_{2}}{(A_{2} + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{k}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi}k^{k+1/2}e^{-k}}{\sqrt{2\pi}(k+\mu+1)^{k+\mu+1+1/2}e^{-(k+\mu+1)}}$$

$$\propto k^{-\mu-1}$$

▶ Since μ depends on A_k , details matter...

Original model

Model detail

Analysis A more plausit

nechanism Robustness

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment

Superlinear attachme kernels

References



- ▶ Time for pure excitement: Find asymptotic behavior of n_k given $A_k \to k$ as $k \to \infty$.
- ► For large *k*:

$$n_{k} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{A_{j}}{A_{j} + \mu}$$

$$= \frac{\mu}{A_{k}} \frac{A_{1}}{(A_{1} + \mu)} \frac{A_{2}}{(A_{2} + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{k}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi} k^{k+1/2} e^{-k}}{\sqrt{2\pi} (k+\mu+1)^{k+\mu+1+1/2} e^{-(k+\mu+1)}}$$

$$\propto k^{-\mu-1}$$

Original model

Model detail

more plausible nechanism

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment kernels

References



- ▶ Time for pure excitement: Find asymptotic behavior of n_k given $A_k \to k$ as $k \to \infty$.
- ► For large *k*:

$$n_{k} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{A_{j}}{A_{j} + \mu}$$

$$= \frac{\mu}{A_{k}} \frac{A_{1}}{(A_{1} + \mu)} \frac{A_{2}}{(A_{2} + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{k}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi} k^{k+1/2} e^{-k}}{\sqrt{2\pi} (k+\mu+1)^{k+\mu+1+1/2} e^{-(k+\mu+1)}}$$

$$\propto k^{-\mu-1}$$

▶ Since μ depends on A_k , details matter...

Original model

Introduction Model detail

> Analysis A more plausib

a more plausit nechanism Robustness

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment kernels

Superlinear attachme kernels

References



- ▶ Time for pure excitement: Find asymptotic behavior of n_k given $A_k \to k$ as $k \to \infty$.
- ► For large *k*:

$$n_{k} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}} = \frac{\mu}{A_{k}} \prod_{j=1}^{k} \frac{A_{j}}{A_{j} + \mu}$$

$$= \frac{\mu}{A_{k}} \frac{A_{1}}{(A_{1} + \mu)} \frac{A_{2}}{(A_{2} + \mu)} \cdots \frac{k-1}{(k-1+\mu)} \frac{k}{(k+\mu)}$$

$$\propto \frac{\Gamma(k)}{\Gamma(k+\mu+1)} \sim \frac{\sqrt{2\pi}k^{k+1/2}e^{-k}}{\sqrt{2\pi}(k+\mu+1)^{k+\mu+1+1/2}e^{-(k+\mu+1)}}$$

$$\propto k^{-\mu-1}$$

▶ Since μ depends on A_k , details matter...

Original model

Model details Analysis

k more plausible nechanism

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment kernels

. .

References



- Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for n_k :

- ▶ Closed form expression for μ .
- ▶ We can solve for μ in some cases.
- ▶ Our assumption that $A = \mu t$ is okay.

Original model

Model detail

A more plausit mechanism

Redner &

Krapivisky's mode

Generalized model

Analysis

Universality?

Sublinear attacl

Superlinear attach

References

Frame 45/57



- Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for n_k :

- ▶ Closed form expression for μ .
- ▶ We can solve for μ in some cases.
- ▶ Our assumption that $A = \mu t$ is okay.

Original model

Model detail

Analysis
A more plausi

Robustness
Redner &

Krapivisky's mode

Analysis Universality?

Sublinear attac

Superlinear attach

References

Frame 45/57



- Now we need to find μ.
- Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for n_k :

- \triangleright Closed form expression for μ .
- We can solve for μ in some cases.
- ightharpoonup Our assumption that $A = \mu t$ is okay.

Redner &

Universality?

References





- Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t)A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for n_k :

- ▶ Closed form expression for μ .
- ▶ We can solve for μ in some cases.
- ▶ Our assumption that $A = \mu t$ is okay.

Original model

Model detail

A more plausit mechanism

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

ernels

References

Frame 45/57



- Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for n_k :

$$\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{i=1}^k \frac{1}{1 + \frac{\mu}{A_i}} A_k$$

- ▶ Closed form expression for μ .
- ▶ We can solve for μ in some cases.
- ▶ Our assumption that $A = \mu t$ is okay.

Original model

Model detail

Analysis
A more plausi
mechanism

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment

ernels

References

Frame 45/57



- ▶ Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t)A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for n_k :

$$\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- ▶ Closed form expression for μ .
- ▶ We can solve for μ in some cases.
- ▶ Our assumption that $A = \mu t$ is okay.

Original model

Model detai

Analysis
A more plausi

Redner &

Analysis
Universality?
Sublinear attachment

.....

References





- ▶ Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for n_k :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- ▶ Closed form expression for μ .
- ▶ We can solve for μ in some cases.
- ▶ Our assumption that $A = \mu t$ is okay.

Model detail

A more plausi mechanism

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

uperlinear attachment ernels

References

Frame 45/57



- Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for n_k :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- ▶ Closed form expression for μ .
- ▶ We can solve for μ in some cases.
- ▶ Our assumption that $A = \mu t$ is okay.

Original model

Model detai

A more plausil mechanism

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

References





- ▶ Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for n_k :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- ▶ Closed form expression for μ .
- ▶ We can solve for μ in some cases.
- ▶ Our assumption that $A = \mu t$ is okay.

Model details
Analysis

Redner &

Generalized model Analysis Universality? Sublinear attachment kernels

References

Frame 45/57



- ▶ Now we need to find μ .
- ▶ Our assumption again: $A = \mu t = \sum_{k=1}^{\infty} N_k(t) A_k$
- Since $N_k = n_k t$, we have the simplification $\mu = \sum_{k=1}^{\infty} n_k A_k$
- Now substitute in our expression for n_k :

$$1\mu = \sum_{k=1}^{\infty} \frac{\mu}{A_k} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}} A_k$$

- ▶ Closed form expression for μ .
- ▶ We can solve for μ in some cases.
- ▶ Our assumption that $A = \mu t$ is okay.

Original model

Model details Analysis

A more plausi mechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

References

Frame 45/57



- ightharpoonup Amazingly, we can adjust A_k and tune γ to be anywhere in $[2, \infty)$.
- $ightharpoonup \gamma = 2$ is the lower limit since

$$\mu = \sum_{k=1}^{\infty} A_k n_k \sim \sum_{k=1}^{\infty} k n_k$$

Let's now look at a specific example of A_k to see this

Original model

Redner &

Universality?

References

Frame 46/57



- ightharpoonup Amazingly, we can adjust A_k and tune γ to be anywhere in $[2, \infty)$.
- ho γ = 2 is the lower limit since

$$\mu = \sum_{k=1}^{\infty} A_k n_k \sim \sum_{k=1}^{\infty} k n_k$$

must be finite.

Let's now look at a specific example of A_k to see this

Original model

Redner &

Universality?

References

Frame 46/57



- ightharpoonup Amazingly, we can adjust A_k and tune γ to be anywhere in $[2, \infty)$.
- ho γ = 2 is the lower limit since

$$\mu = \sum_{k=1}^{\infty} A_k n_k \sim \sum_{k=1}^{\infty} k n_k$$

must be finite.

Let's now look at a specific example of A_k to see this range of γ is possible.

Original model

Redner &

Universality?

References

Frame 46/57



- Find $\gamma = \mu + 1$ by finding μ .
- **Expression** for μ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 - \frac{1}{1 + \frac{\mu}{A_1}} = \frac{1}{1 + \frac{\mu}{A_1}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$\frac{\frac{\mu}{\alpha}}{1+\frac{\mu}{\alpha}} = \frac{1}{1+\frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1+\frac{\mu}{A_j}} \text{ since } A_1 = \alpha$$

Model details

Model detail

A more plaus

mechanism Robustness

Redner & Krapivisky's mode

Analysis
Universality?

Universality? Sublinear attachmen kernels

Superlinear attachm kernels

References



- ▶ Consider $A_1 = \alpha$ and $A_k = k$ for $k \ge 2$.
- Find $\gamma = \mu + 1$ by finding μ .
- \triangleright Expression for μ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 - \frac{1}{1 + \frac{\mu}{A_1}} = \frac{1}{1 + \frac{\mu}{A_1}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$\frac{\frac{\mu}{\alpha}}{1+\frac{\mu}{\alpha}} = \frac{1}{1+\frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1+\frac{\mu}{A_j}} \text{ since } A_1 = \alpha$$

Redner &

Universality?

References



- ▶ Consider $A_1 = \alpha$ and $A_k = k$ for $k \ge 2$.
- Find $\gamma = \mu + 1$ by finding μ .
- **Expression** for μ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 - \frac{1}{1 + \frac{\mu}{A_1}} = \frac{1}{1 + \frac{\mu}{A_1}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$\frac{\frac{\mu}{\alpha}}{1 + \frac{\mu}{\alpha}} = \frac{1}{1 + \frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}} \text{ since } A_1 = \alpha$$

Model deta

Analysis A more plausit

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment kernels

References



- ▶ Consider $A_1 = \alpha$ and $A_k = k$ for $k \ge 2$.
- ▶ Find $\gamma = \mu + 1$ by finding μ .
- **Expression** for μ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 - \frac{1}{1 + \frac{\mu}{A_1}} = \frac{1}{1 + \frac{\mu}{A_1}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$\frac{\frac{\mu}{\alpha}}{1 + \frac{\mu}{\alpha}} = \frac{1}{1 + \frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}} \text{ since } A_1 = \alpha$$

Model deta

Analysis

A more plaus mechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment kernels

References



- ▶ Consider $A_1 = \alpha$ and $A_k = k$ for $k \ge 2$.
- ▶ Find $\gamma = \mu + 1$ by finding μ .
- **Expression** for μ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_1}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$1 - \frac{1}{1 + \frac{\mu}{A_1}} = \frac{1}{1 + \frac{\mu}{A_1}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$\frac{\frac{\mu}{\alpha}}{1+\frac{\mu}{\alpha}} = \frac{1}{1+\frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1+\frac{\mu}{A_j}} \text{ since } A_1 = \alpha$$

Model deta

Analysis

mechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment kernels

References



Redner &

Universality?

References

Universality?

- ▶ Consider $A_1 = \alpha$ and $A_k = k$ for $k \ge 2$.
- ▶ Find $\gamma = \mu + 1$ by finding μ .
- **Expression** for μ :

$$1 = \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}}$$

$$1 = \frac{1}{1 + \frac{\mu}{A_{1}}} + \sum_{k=2}^{\infty} \prod_{j=1}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}}$$

$$1 - \frac{1}{1 + \frac{\mu}{A_{1}}} = \frac{1}{1 + \frac{\mu}{A_{1}}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_{j}}}$$

$$\frac{\frac{\mu}{\alpha}}{1 + \frac{\mu}{\alpha}} = \frac{1}{1 + \frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_{k}}} \text{ since } A_{1} = \alpha$$



► Carrying on:

$$\frac{\frac{\mu}{\alpha}}{\frac{1}{1+\frac{\mu}{\alpha}}} = \frac{1}{1+\frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1+\frac{\mu}{A_j}}$$

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

▶ Now use result that [3]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

with a = 1 and $b = \mu + 1$

$$\mu = \alpha \frac{\Gamma(3)}{(\mu + 1 - 1 - 1)\Gamma(2 + \mu)} \Gamma(2 + \mu)$$

$$\Rightarrow \mu(\mu - 1) = 2\alpha$$

Original model

Model details Analysis

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment
kernels
Superlinear attachment

References

Frame 48/57



Carrying on:

$$\frac{\frac{\mu}{\alpha}}{\frac{1+\frac{\mu}{\alpha}}{\alpha}} = \frac{1}{1+\frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1+\frac{\mu}{A_j}}$$

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

► Now use result that [3]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

with a = 1 and $b = \mu + 1$

$$\mu = \alpha \frac{\Gamma(3)}{(\mu + 1 - 1 - 1)\Gamma(2 + \mu)} \Gamma(2 + \mu)$$

Original model

Model detail

Analysis A more plausib

Redner &

Analysis
Universality?
Sublinear attachment kernels

1101010

References

Frame 48/57



Carrying on:

$$\frac{\frac{\mu}{\alpha}}{\frac{1}{1+\frac{\mu}{\alpha}}} = \frac{1}{1+\frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1+\frac{\mu}{A_j}}$$

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

► Now use result that [3]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

with a = 1 and $b = \mu + 1$.

$$\mu = \alpha \frac{\Gamma(3)}{(\mu + 1 - 1 - 1)\Gamma(2 + \mu)} \Gamma(2 + \mu)$$

 $\Rightarrow \mu(\mu - 1) = 2\alpha$

Original model

Model deta

Analysis A more plausib

mechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment kernels

References

Frame 48/57



Carrying on:

$$\frac{\frac{\mu}{\alpha}}{\frac{1}{1+\frac{\mu}{\alpha}}} = \frac{1}{1+\frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1+\frac{\mu}{A_j}}$$

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

► Now use result that [3]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

with a = 1 and $b = \mu + 1$.

$$\mu = \alpha \frac{\Gamma(3)}{(\mu + 1 - 1 - 1)\Gamma(2 + \mu)} \Gamma(2 + \mu)$$

Original model

Model deta

Analysis A more plausib

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment
kernels
Superlinear attachment

References

Frame 48/57



Carrying on:

$$\frac{\frac{\mu}{\alpha}}{1 + \frac{\mu}{\alpha}} = \frac{1}{1 + \frac{\mu}{\alpha}} \sum_{k=2}^{\infty} \prod_{j=2}^{k} \frac{1}{1 + \frac{\mu}{A_j}}$$

$$\frac{\mu}{\alpha} = \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+\mu+1)}$$

► Now use result that [3]

$$\sum_{k=2}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} = \frac{\Gamma(a+2)}{(b-a-1)\Gamma(b+1)}$$

with a = 1 and $b = \mu + 1$.

>

$$\mu = \alpha \frac{\Gamma(3)}{(\mu + 1 - 1 - 1)\Gamma(2 + \mu)} \Gamma(2 + \mu)$$
$$\Rightarrow \mu(\mu - 1) = 2\alpha$$

Original model

Model details Analysis

A more plausib mechanism Robustness

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment
kernels
Superlinear attachment

References

Frame 48/57



$$\mu(\mu-1)=2\alpha\Rightarrow\mu=\frac{1+\sqrt{1+8\alpha}}{2}.$$

ightharpoonup Since $\gamma = \mu + 1$, we have

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

Craziness...

Original model

Redner &

Universality?

References

Frame 49/57



$$\mu(\mu-1)=2\alpha\Rightarrow\mu=\frac{1+\sqrt{1+8\alpha}}{2}.$$

▶ Since $\gamma = \mu + 1$, we have

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

Craziness...

Original model

Redner &

Universality?

References

Frame 49/57



$$\mu(\mu-1)=2\alpha\Rightarrow\mu=\frac{1+\sqrt{1+8\alpha}}{2}.$$

▶ Since $\gamma = \mu + 1$, we have

$$0 \le \alpha < \infty \Rightarrow 2 \le \gamma < \infty$$

Craziness...

Original model

Redner &

Universality?

References

Frame 49/57



Outline

Model details

Analysis

A more plausible mechanism

Redner & Krapivisky's model

Analysis

Sublinear attachment kernels

Superlinear attachment kernels

Original model

Redner &

Sublinear attachment

References

Frame 50/57



Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with $0 < \nu < 1$.

▶ General finding by Krapivsky and Redner: [3]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- ▶ Universality: now details of kernel do not matter.
- ▶ Distribution of degree is universal providing ν < 1.

Original model

Introduction Model details

Analysis
A more plausil

Redner & Krapivisky's model

Analysis
Universality?
Sublinear attachment

kernels Superlinear attachmen

References



▶ Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with $0 < \nu < 1$.

General finding by Krapivsky and Redner: [3]

$$n_k \sim k^{-
u} e^{-c_1 k^{1-
u} + \text{correction terms}}$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- ▶ Universality: now details of kernel do not matter.
- ▶ Distribution of degree is universal providing ν < 1.

Original model

Introduction Model detail

Analysis
A more plausil

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

ernels Superlinear attachment

References



▶ Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with $0 < \nu < 1$.

General finding by Krapivsky and Redner: [3]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality: now details of kernel do not matter.
- ▶ Distribution of degree is universal providing ν < 1.

Original model

Introduction Model details

Analysis
A more plausi

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

ernels Superlinear attachment

oforonoos

References



Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with $0 < \nu < 1$.

General finding by Krapivsky and Redner: [3]

$$n_k \sim k^{-
u} e^{-c_1 k^{1-
u} + \text{correction terms}}$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality: now details of kernel do not matter.
- ▶ Distribution of degree is universal providing ν < 1.

Original model

Redner &

Sublinear attachment

References



Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with $0 < \nu < 1$.

General finding by Krapivsky and Redner: [3]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality: now details of kernel do not matter.
- ▶ Distribution of degree is universal providing ν < 1.

Original model

Redner &

Sublinear attachment

References



Rich-get-somewhat-richer:

$$A_k \sim k^{\nu}$$
 with $0 < \nu < 1$.

General finding by Krapivsky and Redner: [3]

$$n_k \sim k^{-\nu} e^{-c_1 k^{1-\nu} + \text{correction terms}}$$

- Stretched exponentials (truncated power laws).
- aka Weibull distributions.
- Universality: now details of kernel do not matter.
- ▶ Distribution of degree is universal providing ν < 1.

Original model

Model details
Analysis

Robustness

Redner & Krapivisky's model

Generalized model
Analysis
Universality?
Sublinear attachment

ernels Superlinear attachment Jernels

References



Details:

▶ For $1/2 < \nu < 1$:

$$n_k \sim k^{-\nu} e^{-\mu \left(rac{k^{1-
u}-2^{1-
u}}{1-
u}
ight)}$$

▶ For $1/3 < \nu < 1/2$:

$$n_k \sim k^{-
u} e^{-\mu rac{k^{1-
u}}{1-
u} + rac{\mu^2}{2} rac{k^{1-2
u}}{1-2
u}}$$

And for $1/(r+1) < \nu < 1/r$, we have r pieces in exponential.

Original model

Introduction Model detail

Analysis A more plausil

mechanism Robustness

Redner & Krapivisky's model

Analysis

Universality?

Sublinear attachment kernels

uperiinear attachment ernels

References

Frame 52/57



Details:

▶ For $1/2 < \nu < 1$:

$$n_k \sim k^{-\nu} e^{-\mu \left(rac{k^{1-
u}-2^{1-
u}}{1-
u}
ight)}$$

▶ For $1/3 < \nu < 1/2$:

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

▶ And for $1/(r+1) < \nu < 1/r$, we have *r* pieces in

Original model

Redner &

Sublinear attachment

References

Frame 52/57



Details:

▶ For $1/2 < \nu < 1$:

$$n_k \sim k^{-\nu} e^{-\mu \left(rac{k^{1-
u}-2^{1-
u}}{1-
u}
ight)}$$

▶ For $1/3 < \nu < 1/2$:

$$n_k \sim k^{-\nu} e^{-\mu \frac{k^{1-\nu}}{1-\nu} + \frac{\mu^2}{2} \frac{k^{1-2\nu}}{1-2\nu}}$$

And for $1/(r+1) < \nu < 1/r$, we have r pieces in exponential.

Original model

Model deta

Analysis
A more plausi

Robustness
Redner &

Generalized model

Analysis Universality?

Sublinear attachment kernels

uperlinear attachment ernels

References

i telefelices

Frame 52/57



Outline

Original model

Introduction

Model details

Analysis

A more plausible mechanism

Robustness

Redner & Krapivisky's model

Generalized model

Analysis

Universality?

Sublinear attachment kernels

Superlinear attachment kernels

References

Original model

Model detail

A more plausib nechanism

mechanism Robustness

Redner & Krapivisky's mode

Generalized i Analysis

Universality?

Sublinear attachment kernels Superlinear attachment

kernels

References

Frame 53/57





Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with $\nu > 1$.

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost
- For $\nu > 2$, all but a finite # of nodes connect to one

Original model

Redner &

Superlinear attachment

kernels

References



Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with $\nu > 1$.

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost
- For $\nu > 2$, all but a finite # of nodes connect to one

Original model

Redner &

Superlinear attachment

References

kernels



Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with $\nu > 1$.

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes.
- For $\nu > 2$, all but a finite # of nodes connect to one node.

Original model

Model details Analysis A more plausil

Redner & Krapivisky's mode

Analysis
Universality?
Sublinear attachment

Superlinear attachment kernels

References



Rich-get-much-richer:

$$A_k \sim k^{\nu}$$
 with $\nu > 1$.

- Now a winner-take-all mechanism.
- One single node ends up being connected to almost all other nodes.
- For $\nu > 2$, all but a finite # of nodes connect to one node.

Original model

Redner &

Superlinear attachment

References

kernels



References I

- [1] R. Albert, H. Jeong, and A.-L. Barabási. Error and attack tolerance of complex networks. *Nature*, 406:378–382, July 2000. pdf (\boxplus)
- [2] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. Science, 286:509–511, 1999. pdf (⊞)
- [3] P. L. Krapivsky and S. Redner.
 Organization of growing random networks.

 Phys. Rev. E, 63:066123, 2001. pdf (\(\pm\))
- [4] A. J. Lotka.

 The frequency distribution of scientific productivity.

 Journal of the Washington Academy of Science,
 16:317–323, 1926.

Original model
Introduction
Model details
Analysis
A more plausible
mechanism
Robustness
Redner &
Krapivisky's mod
Generalized model
Analysis
Universality?
Sublinear attachment

References

Frame 55/57

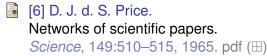


References II



An informational theory of the statistical structure of languages.

In W. Jackson, editor, *Communication Theory*, pages 486–502. Butterworth, Woburn, MA, 1953.



[7] D. J. d. S. Price.

A general theory of bibliometric and other cumulative advantage processes.

J. Amer. Soc. Inform. Sci., 27:292-306, 1976.

[8] H. A. Simon.

On a class of skew distribution functions.

Biometrika, 42:425–440, 1955. pdf (⊞)

Original model

ntroduction Model details Analysis A more plausible mechanism Robustness

Redner & Krapivisky's model

Universality?
Sublinear attach
kernels

kernels

References

Frame 56/57



A mathematical theory of evolution, based on the conclusions of Dr J. C. Willis, F.R.S. Phil. Trans. B, 213:21-, 1924.

[10] G. K. Zipf.

Human Behaviour and the Principle of Least-Effort. Addison-Wesley, Cambridge, MA, 1949.

Original model

Redner &

References

Frame 57/57

