Random Networks

Complex Networks, Course 303A, Spring, 2009

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University of Vermont

Outline

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Random networks

Pure, abstract random networks:

- Consider set of all networks with N labelled nodes and m edges.
- Standard random network = randomly chosen network from this set.
- To be clear: each network is equally probable.
- Sometimes equiprobability is a good assumption, but it is always an assumption.
- Known as Erdős-Rényi random networks or ER graphs.

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Random networks

Some features:

- Number of possible edges:

$$0 \leq m \leq \binom{N}{2} = \frac{N(N-1)}{2}$$

- Given m edges, there are $\binom{\binom{N}{2}}{m}$ different possible networks.
- Crazy factorial explosion for $1 \ll m \ll \binom{N}{2}$.
- Limit of $m = 0$: empty graph.
- Limit of $m = \binom{N}{2}$: complete or fully-connected graph.
- Real world: links are usually costly so real networks are almost always sparse.

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Basics

Random networks

How to build standard random networks:

- Given N and m.
- Two probablistic methods (we'll see a third later on)

1. Connect each of the $\binom{N}{2}$ pairs with appropriate probability p.
   - Useful for theoretical work.
2. Take N nodes and add exactly m links by selecting edges without replacement.
   - Algorithm: Randomly choose a pair of nodes i and j, $i \neq j$, and connect if unconnected; repeat until all m edges are allocated.
   - Best for adding small numbers of links (most cases).
   - 1 and 2 are effectively equivalent for large N.

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A few more things:

- For method 1, # links is probablistic:

$$\langle m \rangle = p\binom{N}{2} = p\frac{1}{2}N(N-1)$$

- So the expected or average degree is

$$\langle k \rangle = \frac{2\,\langle m \rangle}{N}$$

$$= \frac{2}{N}p\frac{1}{2}N(N-1) = \frac{\cancel{2}}{\cancel{N}}p\frac{1}{\cancel{2}}\cancel{N}(N-1) = p(N-1).$$

- Which is what it should be...
- If we keep $\langle k \rangle$ constant then $p \propto 1/N \to 0$ as $N \to \infty$.

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Random networks: examples

Next slides:

Example realizations of random networks

- $N = 500$
- Vary m, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.

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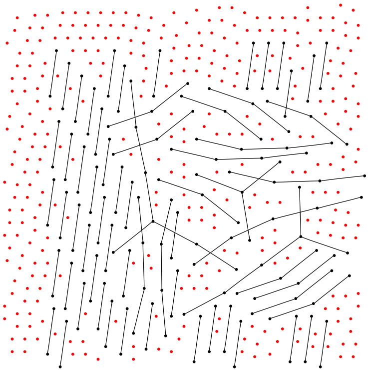
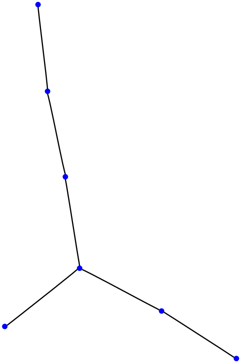
entire network: largest component:

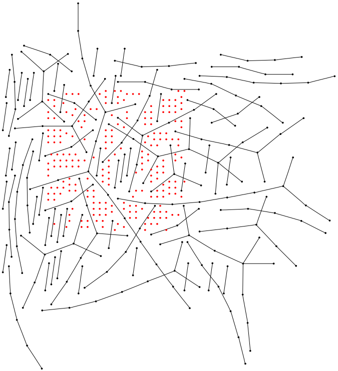




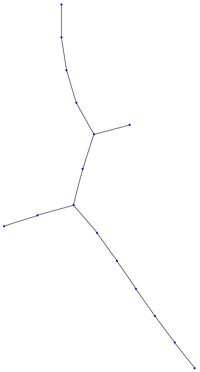
$N = 500$, number of edges $m = 100$
average degree $\langle k \rangle = 0.4$

Random networks: examples

entire network:



largest component:



$N = 500$, number of edges $m = 200$
average degree $\langle k \rangle = 0.8$

Random networks: examples

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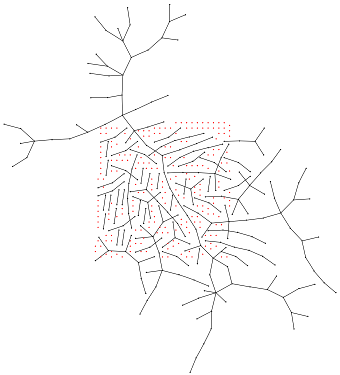

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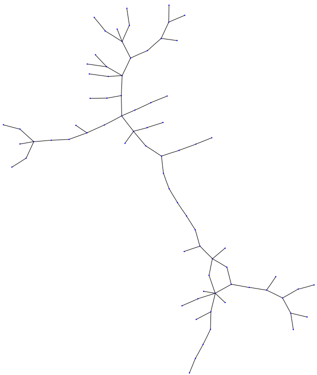
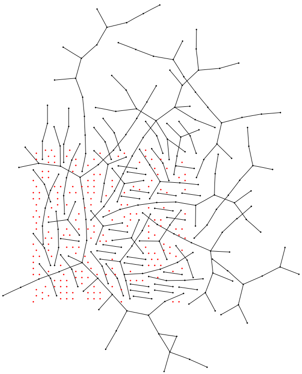


$N = 500$, number of edges $m = 230$
average degree $\langle k \rangle = 0.92$

Random networks: examples

entire network:



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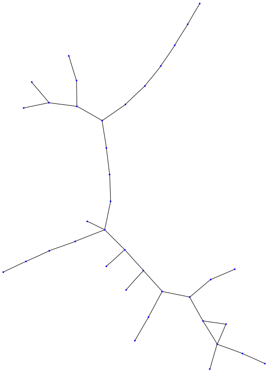
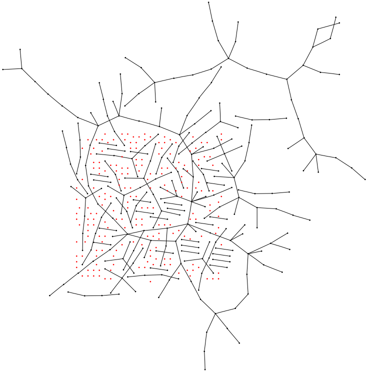


$N = 500$, number of edges $m = 240$
average degree $\langle k \rangle = 0.96$

Random networks: examples

entire network:



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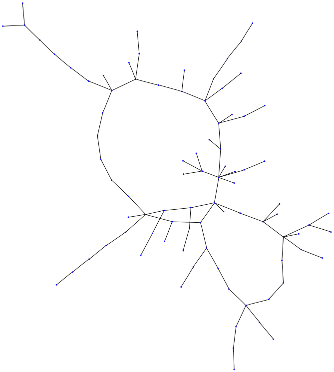
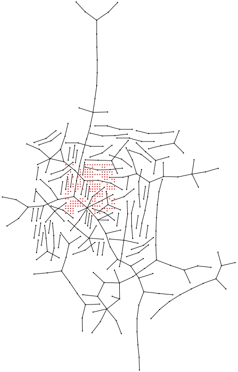


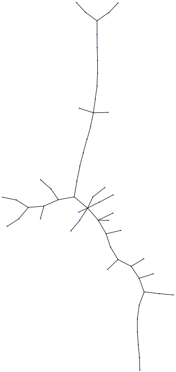
$N = 500$, number of edges $m = 250$
average degree $\langle k \rangle = 1$

Random networks: examples

entire network:



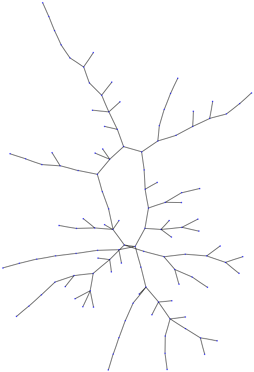
largest component:



$N = 500$, number of edges $m = 260$
average degree $\langle k \rangle = 1.04$

Random networks: examples

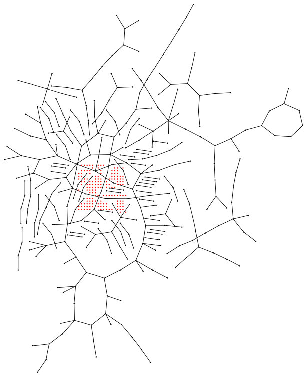
entire network: largest component:



$N = 500$, number of edges $m = 280$
average degree $\langle k \rangle = 1.12$

Random networks: examples

entire network:



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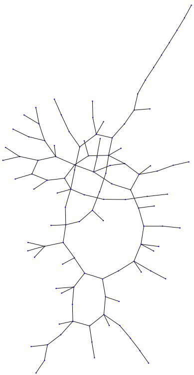
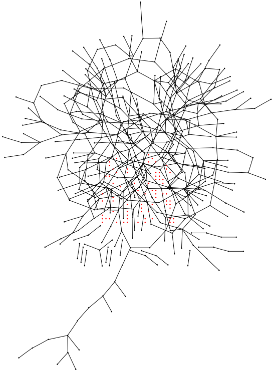


$N = 500$, number of edges $m = 300$
average degree $\langle k \rangle = 1.2$

Random networks: examples

entire network:



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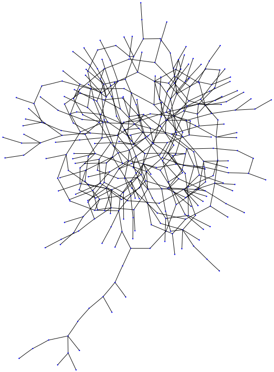
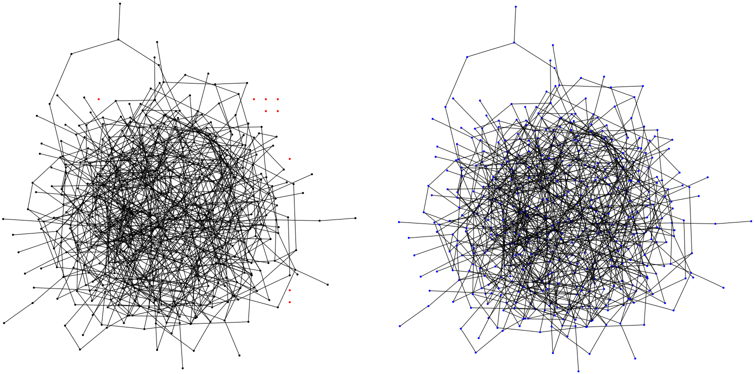


$N = 500$, number of edges $m = 500$
average degree $\langle k \rangle = 2$

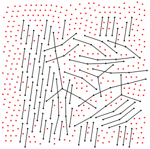
Random networks: examples

entire network: largest component:

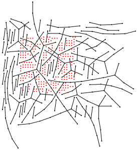


$N = 500$, number of edges $m = 1000$
average degree $\langle k \rangle = 4$

Random networks: examples for N=500

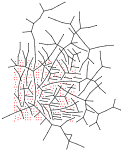


m = 100
$\langle k \rangle$ = 0.4



m = 200
$\langle k \rangle$ = 0.8

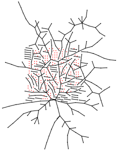
m = 230
$\langle k \rangle$ = 0.92



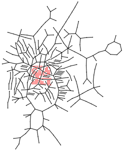
m = 240
$\langle k \rangle$ = 0.96

m = 250
$\langle k \rangle$ = 1

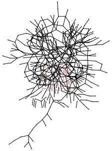
m = 260
$\langle k \rangle$ = 1.04



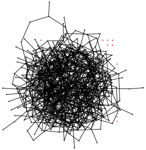
m = 280
$\langle k \rangle$ = 1.12



m = 300
$\langle k \rangle$ = 1.2

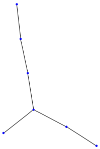


m = 500
$\langle k \rangle$ = 2

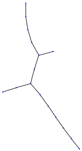


m = 1000
$\langle k \rangle$ = 4

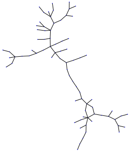
Random networks: largest components



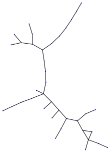
$m = 100$
$\langle k \rangle = 0.4$



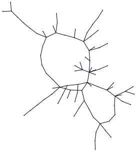
$m = 200$
$\langle k \rangle = 0.8$



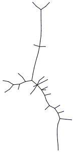
$m = 230$
$\langle k \rangle = 0.92$



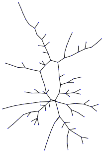
$m = 240$
$\langle k \rangle = 0.96$



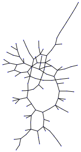
$m = 250$
$\langle k \rangle = 1$



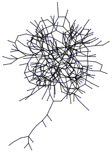
$m = 260$
$\langle k \rangle = 1.04$



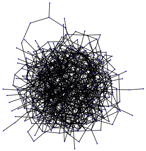
$m = 280$
$\langle k \rangle = 1.12$



$m = 300$
$\langle k \rangle = 1.2$

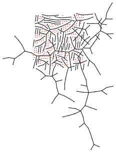


$m = 500$
$\langle k \rangle = 2$



$m = 1000$
$\langle k \rangle = 4$

Random networks: examples for N=500



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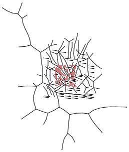

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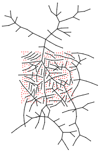
$\langle k \rangle = 1$



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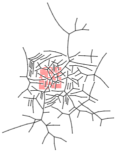
$\langle k \rangle = 1$



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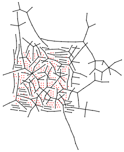

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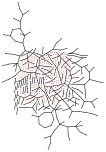

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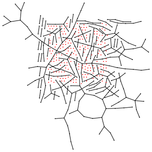

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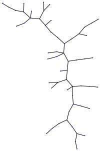
Random networks: largest components



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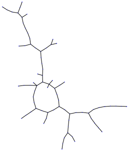

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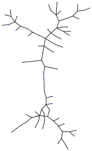

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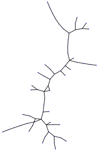

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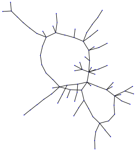

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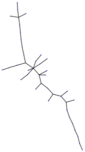

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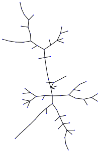

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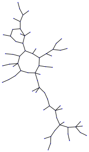

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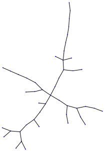

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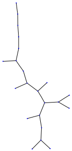

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Outline

Random networks

Clustering:

- For method 1, what is the clustering coefficient for a finite network?
- Consider triangle/triple clustering coefficient (Newman [1]):

$$C_2 = \frac{3 \times \#\text{triangles}}{\#\text{triples}}$$

- Recall: C_2 = probability that two nodes are connected given they have a friend in common.
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Basics
Definitions
How to build
Some visual examples
Structure
Clustering
Degree distributions
Configuration model
Largest component
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
Average Component Size
References

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Degree distribution:

- Recall p_k = probability that a randomly selected node has degree k.
- Consider method 1 for constructing random networks: each possible link is realized with probability p.
- Now consider one node: there are '$N-1$ choose k' ways the node can be connected to k of the other $N-1$ nodes.
- Each connection occurs with probability p, each non-connection with probability $(1-p)$.
- Therefore have a binomial distribution:

$$P(k;p,N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$

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Limiting form of $P(k; p, N)$:

- Our degree distribution: $P(k; p, N) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$.
- What happens as $N \to \infty$?
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- If p is fixed, then we would end up with a Gaussian with average degree $\langle k \rangle \simeq pN \to \infty$.
- But we want to keep $\langle k \rangle$ fixed...
- So examine limit of $P(k; p, N)$ when $p \to 0$ and $N \to \infty$ with $\langle k \rangle = p(N-1) = \text{constant}$.

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- Now use the excellent result:

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

(Use l'Hôpital's rule to prove.)

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Limiting form of $P(k; p, N)$:

- We are now here:

$$P(k; p, N) \simeq \frac{\langle k \rangle^k}{k!} \left(1 - \frac{\langle k \rangle}{N-1}\right)^{N-1-k}$$

- Now use the excellent result:

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

(Use l'Hôpital's rule to prove.)

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Outline

General random networks

- So... standard random networks have a Poisson degree distribution
- Generalize to arbitrary degree distribution P_k.
- Also known as the configuration model [1].
- Can generalize construction method from ER random networks.
- Assign each node a weight w from some distribution P_w and form links with probability

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Random networks: examples

Coming up:

Example realizations of random networks with power law degree distributions:

- $N = 1000$.
- $P_k \propto k^{-\gamma}$ for $k \geq 1$.
- Set $P_0 = 0$ (no isolated nodes).
- Vary exponent γ between 2.10 and 2.91.
- Again, look at full network plus the largest component.
- Apart from degree distribution, wiring is random.

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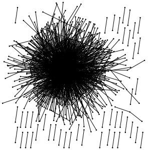

$\gamma = 2.1$
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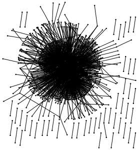

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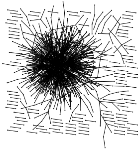

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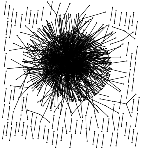

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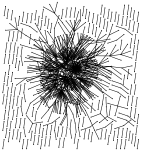

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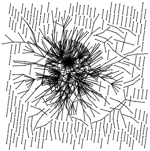

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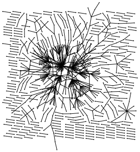

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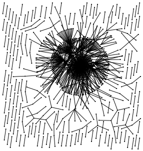

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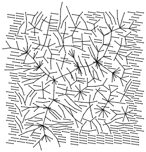

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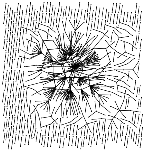

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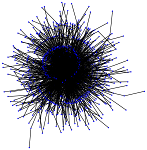
Random networks: largest components



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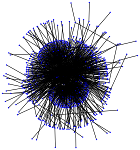

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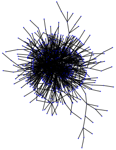

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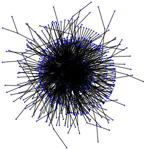

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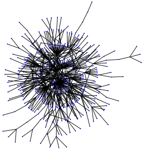

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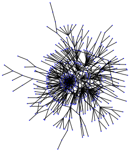

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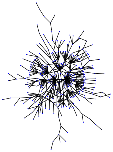

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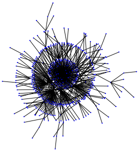

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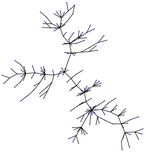

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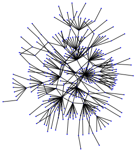

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- Normalization: we must have

$$\sum_{k=0}^{\infty} P(k; \langle k \rangle) = 1$$

- Checking:

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- The variance of degree distributions for random networks turns out to be very important.
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- The degree distribution P_k is fundamental for our description of many complex networks
- Again: P_k is the degree of randomly chosen node.
- A second very important distribution arises from choosing randomly on edges rather than on nodes.
- Define Q_k to be the probability the node at a random end of a randomly chosen edge has degree k.
- Now choosing nodes based on their degree (i.e., size):

$$\boxed{Q_k \propto k P_k}$$

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- Useful variant on Q_k:

  R_k = probability that a friend of a random node has k other friends.

- $$R_k = \frac{(k+1)P_{k+1}}{\sum_{k'=0}(k'+1)P_{k'+1}} = \frac{(k+1)P_{k+1}}{\langle k \rangle}$$
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Two reasons why this matters

Reason #1:

- Average # friends of friends per node is

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- Key: Average depends on the 1st and 2nd moments of P_k and not just the 1st moment.
- Three peculiarities:
  1. We might guess $\langle k_2 \rangle = \langle k \rangle (\langle k \rangle - 1)$ but it's actually $\langle k(k-1) \rangle$.
  2. If P_k has a large second moment, then $\langle k_2 \rangle$ will be big. (e.g., in the case of a power-law distribution)
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More on peculiarity #3:

- A node's average # of friends: $\langle k \rangle$
- Friend's average # of friends: $\frac{\langle k^2 \rangle}{\langle k \rangle}$
- Comparison:

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle \frac{\langle k^2 \rangle}{\langle k \rangle^2} = \langle k \rangle \frac{\sigma^2 + \langle k \rangle^2}{\langle k \rangle^2} = \langle k \rangle \left(1 + \frac{\sigma^2}{\langle k \rangle^2}\right) \geq \langle k \rangle$$

- So only if everyone has the same degree (variance$= \sigma^2 = 0$) can a node be the same as its friends.
- Intuition: for random networks, the more connected a node, the more likely it is to be chosen as a friend.

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$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle \frac{\langle k^2 \rangle}{\langle k \rangle^2} = \langle k \rangle \frac{\sigma^2 + \langle k \rangle^2}{\langle k \rangle^2} = \langle k \rangle \left(1 + \frac{\sigma^2}{\langle k \rangle^2} \right) \geq \langle k$$

- So only if everyone has the same degree (variance$= \sigma^2 = 0$) can a node be the same as its friends.
- Intuition: for random networks, the more connected a node, the more likely it is to be chosen as a friend.

Two reasons why this matters

More on peculiarity #3:

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(Big) Reason #2:

- $\langle k \rangle_R$ is key to understanding how well random networks are connected together.
- e.g., we'd like to know what's the size of the largest component within a network.
- As $N \to \infty$, does our network have a giant component?
- Defn: Component = connected subnetwork of nodes such that $\exists$ path between each pair of nodes in the subnetwork, and no node outside of the subnetwork is connected to it.
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Outline

Structure of random networks

Giant component:

- A giant component exists if when we follow a random edge, we are likely to hit a node with at least 1 other outgoing edge.
- Equivalently, expect exponential growth in node number as we move out from a random node.
- All of this is the same as requiring $\langle k \rangle_R > 1$.
- Giant component condition (or percolation condition):

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1$$

- Again, see that the second moment is an essential part of the story.
- Equivalent statement: $\langle k^2 \rangle > 2\langle k \rangle$

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Giant component

Standard random networks:

- Recall $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$.
- Condition for giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle - \langle k \rangle}{\langle k \rangle} = \langle k \rangle$$

- Therefore when $\langle k \rangle > 1$, standard random networks have a giant component.
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- Fine example of a continuous phase transition (⊞).
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Random networks with skewed P_k:

- e.g, if $P_k = ck^{-\gamma}$ with $2 < \gamma < 3$ then

$$\langle k^2 \rangle = c \sum_{k=0}^{\infty} k^2 k^{-\gamma}$$

$$\sim \int_{x=0}^{\infty} x^{2-\gamma} \mathrm{d}x$$

$$\propto x^{3-\gamma} \Big|_{x=0}^{\infty} = \infty \quad (\gg \langle k \rangle).$$

- So giant component always exists for these kinds of networks.
- Cutoff scaling is k^{-3}: if $\gamma > 3$ then we have to look harder at $\langle k \rangle_R$.

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Giant component

And how big is the largest component?

- Define S_1 as the size of the largest component.
- Consider an infinite ER random network with average degree $\langle k \rangle$.
- Let's find S_1 with a back-of-the-envelope argument.
- Define δ as the probability that a randomly chosen node does not belong to the largest component.
- Simple connection: $\delta = 1 - S_1$.
- Dirty trick: If a randomly chosen node is not part of the largest component, then none of its neighbors are.
- So

$$\delta = \sum_{k=0}^{\infty} P_k \delta^k$$

- Substitute in Poisson distribution...

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$$= e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{(\langle k \rangle \delta)^k}{k!}$$

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- We can figure out some limits and details for $S_1 = 1 - e^{-\langle k \rangle S_1}$.
- First, we can write $\langle k \rangle$ in terms of S_1:

$$\langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}.$$

- As $\langle k \rangle \to 0$, $S_1 \to 0$.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$.
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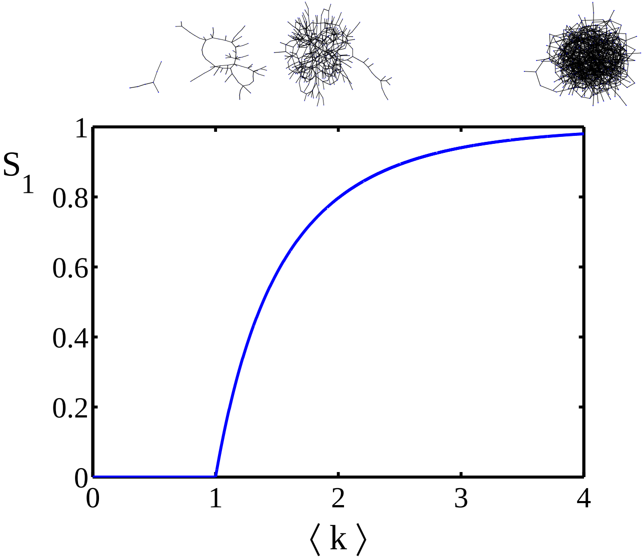

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Turns out we were lucky...

- Our dirty trick only works for ER random networks.
- The problem: We assumed that neighbors have the same probability δ of belonging to the largest component.
- But we know our friends are different from us...
- Works for ER random networks because $\langle k \rangle = \langle k \rangle_R$.
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Outline

Generating functions

- Idea: Given a sequence $a_0, a_1, a_2, \ldots,$ associate each element with a distinct function or other mathematical object.
- Well-chosen functions allow us to manipulate sequences and retrieve sequence elements.

Definition:

- The generating function (g.f.) for a sequence $\{a_n\}$ is

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- Roughly: transforms a vector in R^∞ into a function defined on R^1.
- Related to Fourier, Laplace, Mellin, ...

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Simple example

Rolling dice:

- $p_k^{(\square)} = \mathbf{Pr}(\text{throwing a } k) = 1/6$ where $k = 1, 2, \ldots, 6$.
- $$F^{(\square)}(x) = \sum_{k=1}^{6} p_k x^k = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$
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Example

- Take a degree distribution with exponential decay:

$$P_k = ce^{-\lambda k}$$

where $c = 1 - e^{-\lambda}$.

- The generating function for this distribution is

$$F(x) = \sum_{k=0}^{\infty} P_k x^k = \sum_{k=0}^{\infty} ce^{-\lambda k} x^k = \frac{c}{1 - xe^{-\lambda}}.$$

- Notice that $F(1) = c/(1 - e^{-\lambda}) = 1$.
- For probability distributions, we must always have $F(1) = 1$ since

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Outline

Properties of generating functions

- Average degree:

$$\langle k \rangle = \sum_{k=0}^{\infty} k P_k = \sum_{k=0}^{\infty} k P_k x^{k-1} \Big|_{x=1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} F(x) \Big|_{x=1} = F'(1)$$

- In general, many calculations become simple, if a little abstract.
- For our exponential example:

$$F'(x) = \frac{(1 - e^{-\lambda}) e^{-\lambda}}{(1 - x e^{-\lambda})^2}.$$

- So:

$$\langle k \rangle = F'(1) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})}.$$

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Useful pieces for probability distributions:

- Normalization:

$$F(1) = 1$$

- First moment:

$$\langle k \rangle = F'(1)$$

- Higher moments:

$$\langle k^n \rangle = \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^n F(x) \Big|_{x=1}$$

- kth element of sequence (general):

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Outline

Edge-degree distribution

- Recall our condition for a giant component:

$$\langle k \rangle_R = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} > 1.$$

- Let's rëexpress our condition in terms of generating functions.
- We first need the g.f. for R_k.
- We'll now use this notation:

  $F_P(x)$ is the g.f. for P_k.
  $F_R(x)$ is the g.f. for R_k.

- Condition in terms of g.f. is:

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- Now find how F_R is related to F_P...

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- Now find how F_R is related to F_P...

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- Since we have $F_R(x) = F_P'(x)/F_P'(1)$,

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Size distributions

To figure out the size of the largest component (S_1), we need more resolution on component sizes.

Definitions:

- π_n = probability that a random node belongs to a finite component of size $n < \infty$.
- ρ_n = probability a random link leads to a finite subcomponent of size $n < \infty$.

Local-global connection:

$$P_k, R_k \Leftrightarrow \pi_n, \rho_n$$

neighbors $\Leftrightarrow$ components

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G.f.'s for component size distributions:

- $$F_\pi(x) = \sum_{n=0}^{\infty} \pi_n x^n \text{ and } F_\rho(x) = \sum_{n=0}^{\infty} \rho_n x^n$$

The largest component:

- Subtle key: $F_\pi(1)$ is the probability that a node belongs to a finite component.
- Therefore: $S_1 = 1 - F_\pi(1)$.

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- Find the four generating functions

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Useful results we'll need for g.f.'s

Sneaky Result 1:

- Consider two random variables U and V whose values may be $0, 1, 2, \ldots$
- Write probability distributions as U_k and U_k and g.f.'s as F_U and F_V.
- SR1: If a third random variable is defined as

$$W = \sum_{i=1}^{V} U^{(i)} \text{ with each } U^{(i)} \stackrel{d}{=} U$$

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$$W_k = \sum_{j=0}^{\infty} V_j \times \Pr(\text{sum of } j \text{ draws of variable } U = k)$$

$$= \sum_{j=0}^{\infty} V_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1+i_2+\ldots+i_j=k}} U_{i_1} U_{i_2} \cdots U_{i_j}$$

$$\therefore F_W(x) = \sum_{k=0}^{\infty} W_k x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} V_j \sum_{\substack{\{i_1, i_2, \ldots, i_j\} | \\ i_1+i_2+\ldots+i_j=k}} U_{i_1} U_{i_2} \cdots U_{i_j} x^k$$

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Proof of SR1:

With some concentration, observe:

$$F_W(x) = \sum_{j=0}^{\infty} V_j \sum_{k=0}^{\infty} \underbrace{\sum_{\substack{\{i_1,i_2,\ldots,i_j\}| \\ i_1+i_2+\ldots+i_j=k}} U_{i_1} x^{i_1} U_{i_2} x^{i_2} \cdots U_{i_j} x^{i_j}}_{x^k \text{ piece of } \left(\sum_{i'=0}^{\infty} U_{i'} x^{i'}\right)^j}$$

$$= \sum_{j=0}^{\infty} V_j \left(F_U(x)\right)^j$$

$$= F_V\left(F_U(x)\right) \checkmark$$

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Useful results we'll need for g.f.'s

Sneaky Result 2:

- Start with a random variable U with distribution U_k $(k = 0, 1, 2, \ldots)$
- SR2: If a second random variable is defined as

$$V = U + 1 \text{ then } \boxed{F_V(x) = xF_U(x)}$$

- Reason: $V_k = U_{k-1}$ for $k \geq 1$ and $V_0 = 0$.
- $$\therefore F_V(x) = \sum_{k=0}^{\infty} V_k x^k = \sum_{k=1}^{\infty} U_{k-1} x^k$$

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Generalization of SR2:

- (1) If $V = U + i$ then

$$F_V(x) = x^i F_U(x).$$

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$$F_V(x) = x^{-i} F_U(x)$$

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Outline

Connecting generating functions

- **Goal:** figure out forms of the component generating functions, F_π and F_ρ.
- π_n = probability that a random node belongs to a finite component of size n

$$= \sum_{k=0}^{\infty} P_k \times \Pr\left(\begin{array}{c}\text{sum of sizes of subcomponents}\\ \text{at end of } k \text{ random links} = n-1\end{array}\right)$$

- Therefore: $\boxed{F_\pi(x) = \underbrace{x}_{SR2} \underbrace{F_P\left(F_\rho(x)\right)}_{SR1}}$
- Extra factor of x accounts for random node itself.

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- Invoke one step of recursion: ρ_n = probability that a random node arrived along a random edge is part of a finite subcomponent of size n.

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- We now have two functional equations connecting our generating functions:

$$F_\pi(x) = xF_P(F_\rho(x)) \text{ and } F_\rho(x) = xF_R(F_\rho(x))$$

- Taking stock: We know $F_P(x)$ and $F_R(x) = F_P'(x)/F_P'(1)$.
- We first untangle the second equation to find F_ρ
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Component sizes

- Remembering vaguely what we are doing:

  Finding F_π to obtain the fractional size of the largest component $S_1 = 1 - F_\pi(1)$.

- Set $x = 1$ in our two equations:

$$F_\pi(1) = F_P\left(F_\rho(1)\right) \text{ and } F_\rho(1) = F_R\left(F_\rho(1)\right)$$

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Component sizes

Example: Standard random graphs.

- We can show $F_P(x) = e^{-\langle k \rangle (1-x)}$

$$\therefore F_R(x) = F_P'(x)/F_P'(1) = e^{-\langle k \rangle (1-x)} / e^{-\langle k \rangle (1-x')}\big|_{x'=1}$$

$$= e^{-\langle k \rangle (1-x)} = F_P(x) \qquad \text{...aha!}$$

- RHS's of our two equations are the same.
- So $F_\pi(x) = F_\rho(x) = xF_R(F_\rho(x)) = xF_R(F_\pi(x))$
- Why our dirty (but wrong) trick worked earlier...

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- We are down to $F_\pi(x) = xF_R(F_\pi(x))$ and $F_R(x) = e^{-\langle k \rangle (1-x)}$.
- $$\therefore F_\pi(x) = xe^{-\langle k \rangle (1-F_\pi(x))}$$
- We're first after $S_1 = 1 - F_\pi(1)$ so set $x = 1$ and replace $F_\pi(1)$ by $1 - S_1$:

$$1 - S_1 = e^{-\langle k \rangle S_1}$$

$$\text{Or: } \langle k \rangle = \frac{1}{S_1} \ln \frac{1}{1 - S_1}$$

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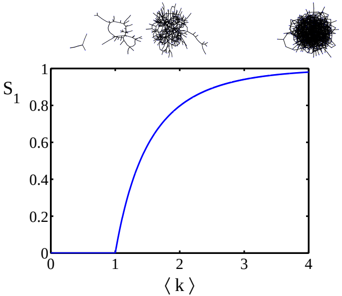

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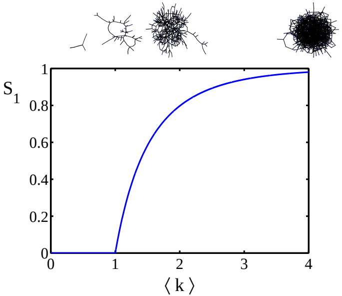

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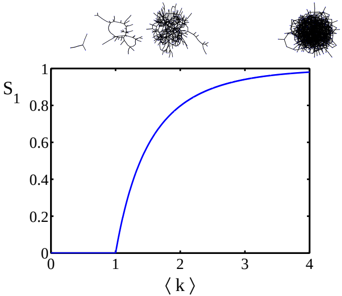

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Outline

Basics
Definitions
How to build
Some visual examples
Structure
Clustering
Degree distributions
Configuration model
Largest component
Generating Functions
Definitions
Basic Properties
Giant Component Condition
Component sizes
Useful results
Size of the Giant Component
Average Component Size
References

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- Next: find average size of finite components $\langle n \rangle$.
- Using standard G.F. result: $\langle n \rangle = F'_\pi(1)$.
- Try to avoid finding $F_\pi(x)$...
- Starting from $F_\pi(x) = xF_P\left(F_\rho(x)\right)$, we differentiate:

$$F'_\pi(x) = F_P\left(F_\rho(x)\right) + xF'_\rho(x)F'_P\left(F_\rho(x)\right)$$

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- Now set $x = 1$ in both equations.
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- Our result for standard random networks:

$$\langle n \rangle = F'_\pi(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

$$\langle n \rangle = \frac{1}{1 - \langle k \rangle}$$

- This blows up as $\langle k \rangle \to 1$.
- Reason: we have a power law distribution of component sizes at $\langle k \rangle = 1$.
- Typical critical point behavior....

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- Typical critical point behavior....

Average component size

- Our result for standard random networks:

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- Recall that $\langle k \rangle = 1$ is the critical value of average degree for standard random networks.
- Look at what happens when we increase $\langle k \rangle$ to 1 from below.
- We have $S_1 = 0$ for all $\langle k \rangle < 1$ so

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- Limits of $\langle k \rangle = 0$ and ∞ make sense for

$$\langle n \rangle = F'_{\pi}(1) = \frac{(1 - S_1)}{1 - \langle k \rangle (1 - S_1)}$$

- As $\langle k \rangle \to 0$, $S_1 = 0$, and $\langle n \rangle \to 1$.
- All nodes are isolated.
- As $\langle k \rangle \to \infty$, $S_1 \to 1$ and $\langle n \rangle \to 0$.
- No nodes are outside of the giant component.

Extra on largest component size:

- For $\langle k \rangle = 1$, $S_1 \sim N^{2/3}$.
- For $\langle k \rangle < 1$, $S_1 \sim \log N$.

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