Measures of centrality

Complex Networks, Course 303A, Spring, 2009

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont

Outline

How big is my node?

- Basic question: how 'important' are specific nodes and edges in a network?
- An important node or edge might:
  1. handle a relatively large amount of the network's traffic (e.g., cars, information);
  2. bridge two or more distinct groups (e.g., liason, interpreter);
  3. be a source of important ideas, knowledge, or judgments (e.g., supreme court decisions, an employee who 'knows where everything is').
- So how do we quantify such a slippery concept as importance?
- We generate ad hoc, reasonable measures, and examine their utility...

Centrality

- One possible reflection of importance is centrality.
- Presumption is that nodes or edges that are (in some sense) in the middle of a network are important for the network's function.
- Idea of centrality comes from social networks literature [7].
- Many flavors of centrality...
  1. Many are topological and quasi-dynamical;
  2. Some are based on dynamics (e.g., traffic).
- We will define and examine a few...
- (Later: see centrality useful in identifying communities in networks.)

Centrality

Degree centrality

- Naively estimate importance by node degree. [7]
- Doh: assumes linearity
  (If node i has twice as many friends as node j, it's twice as important.)
- Doh: doesn't take in any non-local information.

Closeness centrality

- Idea: Nodes are more central if they can reach other nodes 'easily.'
- Measure average shortest path from a node to all other nodes.
- Define Closeness Centrality for node i as

$$\frac{N-1}{\sum_{j, j\neq i}(\text{distance from } i \text{ to } j).}$$

- Range is 0 (no friends) to 1 (single hub).
- Unclear what the exact values of this measure tells us because of its ad-hocness.
- General problem with simple centrality measures: what do they exactly mean?
- Perhaps, at least, we obtain an ordering of nodes in terms of 'importance.'

Betweenness centrality

- Betweenness centrality is based on shortest paths in a network.
- Idea: If the quickest way between any two nodes on a network disproportionately involves certain nodes, then they are ‘important’ in terms of global cohesion.
- For each node i, count how many shortest paths pass through i.
- In the case of ties, or divide counts between paths.
- Call frequency of shortest paths passing through node i the betweenness of i, B_i.
- Note: Exclude shortest paths between i and other nodes.
- Note: works for weighted and unweighted networks.

- Consider a network with N nodes and m edges (possibly weighted).
- Computational goal: Find $\binom{N}{2}$ shortest paths (⊞) between all pairs of nodes.
- Traditionally use Floyd-Warshall (⊞) algorithm.
- Computation time grows as $O(N^3)$.
- See also:
  1. Dijkstra's algorithm (⊞) for finding shortest path between two specific nodes,
  2. and Johnson's algorithm (⊞) which outperforms Floyd-Warshall for sparse networks: $O(mN + N^2 \log N)$.
- Newman (2001) [4, 5] and Brandes (2001) [1] independently derive much faster algorithms.
- Computation times grow as:
  1. $O(mN)$ for unweighted graphs;
  2. and $O(mN + N^2 \log N)$ for weighted graphs.

Shortest path between node *i* and all others:

- Consider unweighted networks.
- Use breadth-first search:
  1. Start at node i, giving it a distance $d = 0$ from itself.
  2. Create a list of all of i's neighbors and label them being at a distance $d = 1$.
  3. Go through list of most recently visited nodes and find all of their neighbors.
  4. Exclude any nodes already assigned a distance.
  5. Increment distance d by 1.
  6. Label newly reached nodes as being at distance d.
  7. Repeat steps 3 through 6 until all nodes are visited.
- Record which nodes link to which nodes moving out from i (former are 'predecessors' with respect to i's shortest path structure).
- Runs in $O(m)$ time and gives N shortest paths.
- Find all shortest paths in $O(mN)$ time
- Much, much better than naive estimate of $O(mN^2)$.

Newman's Betweenness algorithm: [4]

1. Set all nodes to have a value $c_{ij} = 0$, $j = 1, ..., N$ (c for count).
2. Select one node i.
3. Find shortest paths to all other $N - 1$ nodes using breadth-first search.
4. Record # equal shortest paths reaching each node.
5. Move through nodes according to their distance from i, starting with the furthest.
6. Travel back towards i from each starting node j, along shortest path(s), adding 1 to every value of $c_{i\ell}$ at each node ℓ along the way.
7. Whenever more than one possibility exists, apportion according to total number of short paths coming through predecessors.
8. Exclude starting node j and i from increment.
9. Repeat steps 2–8 for every node i and obtain betweenness as $B_j = \sum_{i=1}^{N} c_{ij}$.

Newman's Betweenness algorithm: [4]

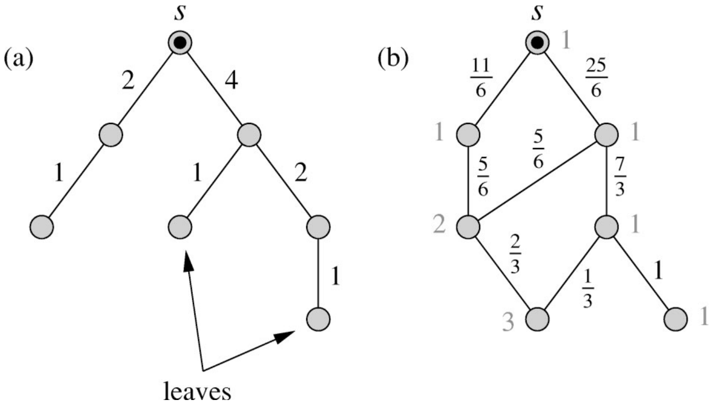
- For a pure tree network, c_{ij} is the number of nodes beyond j from i's vantage point.
- Same algorithm for computing drainage area in river networks (with 1 added across the board).
- For edge betweenness, use exact same algorithm but now
  1. j indexes edges,
  2. and we add one to each edge as we traverse it.
- For both algorithms, computation time grows as

$$O(mN).$$

- For sparse networks with relatively small average degree, we have a fairly digestible time growth of

$$O(N^2).$$

Newman's Betweenness algorithm: [4]



Important nodes have important friends:

- Define x_i as the 'importance' of node i.
- Idea: x_i depends (somehow) on x_j
  if j is a neighbor of i.
- Recursive: importance is transmitted through a network.
- Simplest possibility is a linear combination:

$$x_i \propto \sum_j a_{ji} x_j$$

- Assume further that constant of proportionality, c, is independent of i.
- Above gives $\vec{x} = c\mathbf{A}^{\mathrm{T}}\vec{x}$ or $\boxed{\mathbf{A}^{\mathrm{T}}\vec{x} = c^{-1}\vec{x} = \lambda\vec{x}}$.
- Eigenvalue equation based on adjacency matrix...
- Note: Lots of despair over size of the largest eigenvalue. [7] Lose sight of original assumption's non-physicality.

Important nodes have important friends:

- So... solve $\mathbf{A}^{\mathrm{T}}\vec{x} = \lambda\vec{x}$.
- But which eigenvalue and eigenvector?
- We, the people, would like:
  1. A unique solution. ✓
  2. λ to be real. ✓
  3. Entries of $\vec{x}$ to be real. ✓
  4. Entries of $\vec{x}$ to be non-negative. ✓
  5. λ to actually mean something... (maybe too much)
  6. Values of x_i to mean something (what does an observation that $x_3 = 5x_7$ mean?) (maybe only ordering is informative...) (maybe too much)
  7. λ to equal 1 would be nice... (maybe too much)
  8. Ordering of $\vec{x}$ entries to be robust to reasonable modifications of linear assumption (maybe too much)
- We rummage around in bag of tricks and pull out the Perron-Frobenius theorem...

Perron-Frobenius theorem: (⊞)

If an $N \times N$ matrix A has non-negative entries then:

1. A has a real eigenvalue $\lambda_1 \geq |\lambda_i|$ for $i = 2, \ldots, N$.
2. λ_1 corresponds to left and right 1-d eigenspaces for which we can choose a basis vector that has non-negative entries.
3. The dominant real eigenvalue λ_1 is bounded by the minimum and maximum row sums of A:

$$\min_i \sum_{j=1}^{N} a_{ij} \leq \lambda_1 \leq \max_i \sum_{j=1}^{N} a_{ij}$$

4. All other eigenvectors have one or more negative entries.
5. The matrix A can make toast.
6. Note: Proof is relatively short for symmetric matrices that are strictly positive [6] and just non-negative [3].

Other Perron-Frobenius aspects:

- Assuming our network is irreducible (⊞), meaning there is only one component, is reasonable: just consider one component at a time if more than one exists.
- Irreducibility means largest eigenvalue's eigenvector has strictly non-negative entries.
- Analogous to notion of ergodicity: every state is reachable.
- (Another term: Primitive graphs and matrices.)

Hubs and Authorities

- Generalize eigenvalue centrality to allow nodes to have two attributes:
  1. Authority: how much knowledge, information, etc., held by a node on a topic.
  2. Hubness (or Hubosity): how well a node 'knows' where to find information on a given topic.
- Original work due to the legendary Jon Kleinberg. [2]
- Best hubs point to best authorities.
- Recursive: nodes can be both hubs and authorities.
- More: look for dense links between sets of good hubs pointing to sets of good authorities.
- Known as HITS algorithm (⊞) (Hyperlink-Induced Topics Search).

Hubs and Authorities

- Give each node two scores:
  1. x_i = authority score for node i
  2. y_i = hubtasticness score for node i
- As for eigenvector centrality, we connect the scores of neighboring nodes.
- New story I: a good authority is linked to by good hubs.
- Means x_i should increase as $\sum_{j=1}^{N} a_{ji} y_j$ increases.
- Note: indices are ji meaning j has a directed link to i.
- New story II: good hubs point to good authorities.
- Means y_i should increase as $\sum_{j=1}^{N} a_{ij} x_j$ increases.
- Linearity assumption:

$$\vec{x} \propto A^T \vec{y} \text{ and } \vec{y} \propto A\vec{x}$$

Hubs and Authorities

- So let's say we have

$$\vec{x} = c_1 A^T \vec{y} \text{ and } \vec{y} = c_2 A \vec{x}$$

  where c_1 and c_2 must be positive.
- Above equations combine to give

$$\vec{x} = c_1 A^T c_2 A \vec{x} = \lambda A^T A \vec{x}.$$

  where $\lambda = c_1 c_2 > 0$.
- It's all good: we have the heart of singular value decomposition before us...

We can do this:

- $A^T A$ is symmetric.
- $A^T A$ is semi-positive definite so its eigenvalues are all ≥ 0.
- $A^T A$'s eigenvalues are the square of A's singular values.
- $A^T A$'s eigenvectors are form a joyful orthogonal basis.
- Perron-Frobenius tells us that only the dominant eigenvalue's eigenvector can be chosen to have non-negative entries.
- So: linear assumption leads to a solvable system.
- What would be very good: find networks where we have independent measures of node 'importance' and see how importance is actually distributed.

References I

[1] U. Brandes.
A faster algorithm for betweenness centrality.
J. Math. Sociol., 25:163–177, 2001. pdf (⊞)

[2] J. M. Kleinberg.
Authoritative sources in a hyperlinked environment.
Proc. 9th ACM-SIAM Symposium on Discrete Algorithms, 1998. pdf (⊞)

[3] K. Y. Lin.
An elementary proof of the perron-frobenius theorem for non-negative symmetric matrices.
Chinese Journal of Physics, 15:283–285, 1977.
pdf (⊞)

References II

[4] M. E. J. Newman.
Scientific collaboration networks. II. Shortest paths, weighted networks, and centrality.
Phys. Rev. E, 64(1):016132, 2001.

[5] M. E. J. Newman and M. Girvan.
Finding and evaluating community structure in networks.
Phys. Rev. E, 69(2):026113, 2004. pdf (⊞)

[6] F. Ninio.
A simple proof of the Perron-Frobenius theorem for positive symmetric matrices.
J. Phys. A.: Math. Gen., 9:1281–1282, 1976. pdf (⊞)

[7] S. Wasserman and K. Faust.
Social Network Analysis: Methods and Applications.
Cambridge University Press, Cambridge, UK, 1994.