Complex Networks, CSYS/MATH 303—Assignment 2
University of Vermont, Spring 2009

Dispersed: Wednesday, February 11, 2009.
Due: By start of lecture, 10:00 am, Tuesday, February 24, 2009.
Some useful reminders:
Instructor: Peter Dodds
Office: 203 Lord House, 16 Colchester Avenue (TR)
Other office: Farrell Hall, 210 Colchester Ave (MWF)
E-mail: peter.dodds@uvm.edu
Office phone: (802) 656-3089 (email is better...)
Office hours: 2:30 pm to 3:30 pm, Tuesday & 11:30 am to 12:30 pm Thursday
Course website: `http://www.uvm.edu/~pdodds/teaching/courses/2009-01UVM-303/`

All parts are worth 3 points unless marked otherwise. Please show all your working clearly and list the names of others with whom you collaborated.

I. Supply networks and allometry:

1. From lectures on Supply Networks:

   Show that for large V

   $$\min V_{\rm net} \propto \int_{\Omega_{d,D}(V)} \rho \,||\vec{x}||^{1-2\epsilon}\, \mathrm{d}\vec{x} \sim \rho V^{1+\gamma_{\rm max}(1-2\epsilon)}$$

   Reminders: we defined $L_i = c_i^{-1} V^{\gamma_i}$ where $\gamma_1 + \gamma_2 + \ldots + \gamma_d = 1$, $\gamma_1 = \gamma_{\rm max} \geq \gamma_2 \geq \ldots \geq \gamma_d$., and $c = \prod_i c_i \leq 1$ is a shape factor.

   Hints: assume the first k lengths scale in the same way with $\gamma_1 = \ldots = \gamma_k = \gamma_{\rm max}$, and write $||\vec{x}|| = (x_1^2 + x_2^2 + \ldots + x_d^2)^{1/2}$.

2. Consider a set of rectangular areas with side lengths L_1 and L_2 such that $L_1 \propto A^{\gamma_1}$ and $L_2 \propto A^{\gamma_2}$ where A is area and $\gamma_1 + \gamma_2 = 1$. Assume $\gamma_1 > \gamma_2$.

   Now imagine that material has to be distributed from a central source in each of these areas to sinks distributed with density $\rho(A)$, and that these sinks draw the same amount of material per unit time independent of L_1 and L_2.

   Find an exact form for how the volume of the most efficient distribution network scales with overall area $A = L_1 L_2$. (Hint: you will have to set up a double integration over the rectangle.)

   If network volume must remain a constant fraction of overall area, determine the maximal scaling of sink density ρ with A.

II. Size-density law:

In two dimensions, the size-density law for distributed source density $D(\vec{x})$ given a sink density $\rho(\vec{x})$ states that $D \propto \rho^{2/3}$. We showed in class that an approximate argument that minimizes the average distance between sinks and nearest sources gives the 2/3 exponent (see Supply Networks lecture notes).

1. Repeat this argument for the d-dimensional case and find the general form of the exponent β in $D \propto \rho^{\beta}$.

2. In 1-d, consider a population density $\rho(x) = cx^{-\gamma}$ for $x \geq 1$ and $\gamma > 2$ (note that $c = \gamma - 1$).

   Find the ideal distribution for N sources where N is large.

   Hint: draw yourself a clear picture of what's going on.

   Hint: guess the form of the locations of the centers and work from there.

   Also: Feel free to do some numerics to see how things work.

3. Repeat the above treatment for $\rho(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.