Chapter 6: Lecture 25

Linear Algebra, Course 124C, Spring, 2009

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Ch. 6: Lec. 25 The fundamental theorem of linear algebra Approximating matrices with SVD The basic idea

All the way with $A\vec{x} = \vec{b}$:

- ▶ Applies to any $m \times n$ matrix A.
- ightharpoonup Symmetry of A and A^{T} .

Where \vec{x} lives:

- ▶ Row space $C(A^T) \subset R^n$.
- ▶ (Right) Nullspace $N(A) \subset R^n$.
- ▶ Orthogonality: $C(A^T) \bigotimes N(A) = R^n$

Where \vec{b} lives:

- ▶ Column space $C(A) \subset R^m$.
- ▶ Left Nullspace $N(A^T) \subset R^m$.
- ▶ dim C(A) + dim $N(A^{T})$ = r + (m r) = m
- ▶ Orthogonality: $C(A) \otimes N(A^{T}) = R^{m}$

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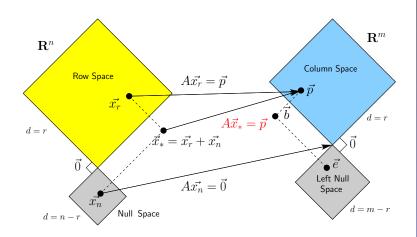
The fundamental theorem of linear algebra

Approximating matrices with SVII

The basic idea



Best solution \vec{x}_* when $\vec{b} = \vec{p} + \vec{e}$:



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Fundamental Theorem of Linear Algebra

Now we see:

- Each of the four fundamental subspaces has a 'best' orthonormal basis
- ▶ The \hat{v}_i span R^n
- ▶ We find the \hat{v}_i as eigenvectors of A^TA .
- ▶ The \hat{u}_i span R^m
- ▶ We find the \hat{u}_i as eigenvectors of AA^T .

Happy bases

- ▶ $\{\hat{v}_1, \dots, \hat{v}_r\}$ span Row space
- $ightharpoonup \{\hat{v}_{r+1}, \dots, \hat{v}_n\}$ span Null space
- $\{\hat{u}_1, \dots, \hat{u}_r\}$ span Column space
- $\{\hat{u}_{r+1}, \dots, \hat{u}_m\}$ span Left Null space

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Fundamental Theorem of Linear Algebra

How $A\vec{x}$ works:

- $ightharpoonup A = U\Sigma V^{\mathrm{T}}$
- ▶ A sends each $\vec{v}_i \in C(A^T)$ to its partner $\vec{u}_i \in C(A)$ with a stretch/shrink factor $\sigma_i > 0$.
- ▶ A is diagonal with respect to these bases and has positive entries (all $\sigma_i > 0$).
- When viewed the right way, any A is a diagonal matrix Σ.

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Image approximation (80x60)

Idea: use SVD to approximate images

- ► Interpret elements of matrix *A* as color values of an image.
- ► Truncate series SVD representation of *A*:

$$A = U\Sigma V^{\mathrm{T}} = \sum_{i=1}^{r} \sigma_{i} \hat{u}_{i} \hat{v}_{i}^{\mathrm{T}}$$

- Use fact that $\sigma_1 > \sigma_2 > \ldots > \sigma_r > 0$.
- ▶ Rank $r = \min(m, n)$.
- ▶ Rank r = # of pixels on shortest side.
- ► For color: approximate 3 matrices (RGB).

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